SNR argument

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1 Definitions

This is an attempt to get the various physical quantities sorted out for SNR considerations for HBT interferometry. Here is what I understand:

Symbol	units	Description
A	m^2 (length^2)	area of collection telescope
η	-	quantum efficiency of photodetector
λ	m (length)	optical wavelength
ν	Hz (1/time)	frequency of a photon
R	-	reflectivity of telescope mirror. This is used to cover various telescope losses.
T	s (time)	total integration time of a measurement
τ_t	s (time)	detector and electronic time resolution, or inverse electrical bandwidth in original HBT measurements
τ_c	s (time)	coherence time of light (before or after filtering, or general)
V	-	Visibility of the first order correlation. For an ideal thermal light source, $V = 1$. For non-ideal cases, one should see $g^{(2)}(\tau = 0) = 1+V^2$ (this is the Glauber 1963 result?).

There are definitions / physical quantities where I believe we have uncertainties:

Symbol	units	Description
n	- (photons/time /frequency)	spectral density for incoming light, expressed in the number of photons per observation time per filter bandwidth. This quantity makes sense for a continuous spectrum. In some places, it appears as if this is referred to the source area of the receiver, specifically in the standard SNR expression.
I	$\mathrm{W}^{-1}\mathrm{m}^{-2}$	Intensity, received power per area

2 Collateral relations

2.1 Blackbody radiation

The spectral radiance of an ideal blackbody is given by Planck's law:

$$B_\nu(\nu) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1} \tag{1}$$

with a unit of $\mathrm{W}^{-1}\mathrm{sr}^{-1}\mathrm{m}^{-2}\mathrm{Hz}^{-1}$. The area refers to thesource area, and the solid angle to the emission direction. For a thermal light source at distance d with source area A_S, we collect into a telescope with an area A optical power with the following spectral intensity (assuming isotropic emission, unit emittivity, i.e. ideal blackbody)

$$I_\nu(\nu) = B_\nu(\nu) A_S \cdot \frac{A}{d^2} \tag{2}$$

The unit of this quantity is $\mathrm{W}^{-1}\mathrm{m}^{-2}\mathrm{Hz}^{-1}$. We probably want to express this not in terms of Watts, but photons per second since we count photons. This leads to the spectral density of a star (in an ideal scenario) of

$$\begin{aligned} n_\nu(\nu) &= I_\nu(\nu)/(h\nu) = && (3)\\ &= B_\nu(\nu) A_S \cdot \frac{A}{d^2} \cdot \frac{1}{h\nu} = && (4)\\ &= \frac{2\nu^2}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1} \cdot \frac{A_S}{d^2} \cdot A && (5) \end{aligned}$$

This quantity describes the number of photons per second and frequency and is therefore dimensionless. It appears that in some of the expressions the area A is taken out of this expression, leading to a relation between n and n_ν:

$$\begin{aligned} n &= n_\nu / A && (6)\\ &= \frac{2}{\lambda^2} \cdot \frac{1}{e^{h\nu/kT} - 1} \cdot \frac{A_S}{d^2} && (7) \end{aligned}$$

3 Referee argument

One starts with a spectral density I_0 of light (Does this correspond to our n?). Then behind a filter that enforces a coherence time τ_c of the light, one ends up with a photon rate

$$I = I_0/\tau_c\,, \tag{8}$$

which has a unit (1/time). This makes sense, since a large coherence time corresponds to a small bandwidth, and for a continuous spectrum, the rate should be proportional to the filter bandwidth.

If we were to deal with uncorrelated light or coherent states, the probability of finding a second photon in a time interval of width τ_c is given by $I \cdot \tau_c$. The total number of photons contributing to the bunching signal is just given by the single photon rate, times the probability to find a second one in a coherence time window of width τ_c. Thus, the rate of pair events contributing to the signal would be

$$r_s = I * I * \tau_c \tag{9}$$

To determine the noise on such a measurement, one needs to observe the number of background pair events for a given coincidence time window (or detection resolution window) of τ_t. This pair rate is given by

$$r_b = I * (I * \tau_t)\,. \tag{10}$$

With this, the signal and background is given by the rates, multiplied by the integration time T. For $\tau_c \ll \tau_t$, the background is much larger than the signal, and the noise is determined by the background completely. With a signal $S = r_s T$ and background $B = r_b T$, and assuming Poissonian noise on the background, one gets

$$\begin{aligned} SNR &= \frac{S}{\sqrt{B}} = \sqrt{T} \cdot I \frac{\tau_c}{\sqrt{\tau_t}} && (11) \\ &= I_0 \sqrt{T} \frac{1}{\sqrt{\tau_t}}\,, && (12) \end{aligned}$$

which is independent of the filtering bandwidth τ_c, and only depends on the detection time resolution τ_t.

4 Standard SNR expression

The "standard" expression for the signal to noise ratio is

$$SNR = An\,\eta R\,V^2 \sqrt{\frac{T}{2\tau_t}} \tag{13}$$

This is compatible with the referee's argument about the SNR, since it has the same dependency on T and τ_t. Also, the fraction of photons that contributes to a signal in a non-ideal case (i.e., limited visibility) scales correctly with V^2. The I_0 in the above argument would also be given by $I_0 = An\eta R$, so expression (13) is consistent with the referee's argument. I am missing the 2 in the square root, probably this comes from the splitting in the BS – but this is a minor problem.

5 Our argument

Since our detector resolution τ_t is much smaller than the coherence time τ_c, we first need to come up with a reasonable definition of what we mean when we talk about signal. I assume that in the argument brought forward, one only looks at the pairs in the single time bin of the histogram where $g^{(2)}(\tau)$ is maximal, i.e., near $\tau = 0$. This is conservative, because we also have information from the areas where $g^{(2)}(\tau)$ still exceeds 1. In a fair argument, we would need to include all this information.

In this window, the bunching signal pair rate following a similar argument than above would use a modified probability to find a second photon, that is given by the uncorrelated probability

$p = I * \tau_t$ of finding a second photon in the time window τ_t and an average of the Lorentzian shape near the center; let's refer to it as some sort of visibility V' that you calculated to be around 0.85 for our situation. This leads to a modified signal rate (i.e., useful pair events) of

$$r_s = I * I * \tau_t * V' \,, \tag{14}$$

whereas the noise rate is in the same way determined by the background rate in (10). Thus, the resulting signal-to-noise rate should be

$$\begin{aligned} SNR_f &= \frac{r_s T}{\sqrt{r_b T}} = \frac{I^2 \tau_t V' T}{\sqrt{I^2 \tau_t T}} && (15) \\ &= I\sqrt{T}\sqrt{\tau_t} V' && (16) \\ &= I_0 \sqrt{T}\sqrt{\frac{\tau_t}{\tau_c^2}} V' && (17) \end{aligned}$$

We can re-express this SNR in (12), and find

$$SNR_f = SNR \cdot V' \frac{\tau_t}{\tau_c} \,, \tag{18}$$

which is *smaller* than the original SNR by the ratio of the time ratios.... somehow this is not yet surprising, because we take a much smaller time window for pair detections, and the signal is proportional to that window, while the noise only grows with the square root. So we have to look how we can improve the SNR by combining all time windows which contain information.

6 Advanced SNR expression

$$SNR = V^2 \frac{\tau_c}{2} \sqrt{N_1 N_2} \sqrt{\frac{T}{2\tau_t}} \tag{19}$$

Therein, $N_{1,2}$ describe the total number of photons detected by each detectors.

7 What is V^2?

For spatial intensity interferometry in astronomy, the Van Cittert-Zernicke theorem is invoked, which states that for an incoherent, quasi-monochromatic source of radiation, **the equal-time** degree of coherence is proportional to the complex amplitude. (any textbook on stellar HBT interferometry discuss this)

In other words, by the nature of the derivation for spatial $g^{(2)}(b)$, the condition is assumed and enforced in theory that: (where b = baseline = spatial separation between detectors)

$$\tau = 0 \tag{20}$$

This simplifies $g^{(2)}(b, \tau)$ to a purely spatial $g^{(2)}(b)$ consideration.

Let us examine the standard accepted SNR expression in literature:

$$SNR_{HBT} = A\eta RI_0\sqrt{\frac{T}{\tau_t}} \cdot V^2(b, \tau = 0) \tag{21}$$

Assuming an ideal interferometer such that the Van Cittert-Zernicke condition $\tau = 0$ can be measured, then it is clear that the visibility term V^2 is purely spatial, scales to 1 for b = 0. However, assuming a real interferometer such that the closest to $\tau = 0$ is an averaged measurement between $\tau = 0$ to $\tau = \tau_t$, then the measured visibility should be the following:

$$V^2(\tau \approx 0, \tau_t, \tau_c) = \frac{1}{\tau_t}\int_{\tau=0}^{\tau=\tau_t} e^{-2\tau/\tau_c} d\tau \tag{22}$$

$$SNR_{ours} = A\eta I_0\sqrt{\frac{T}{\tau_t}} \cdot \frac{\tau_c}{2\tau_t}\left(1 - e^{-2\tau_t/\tau_c}\right) \tag{23}$$

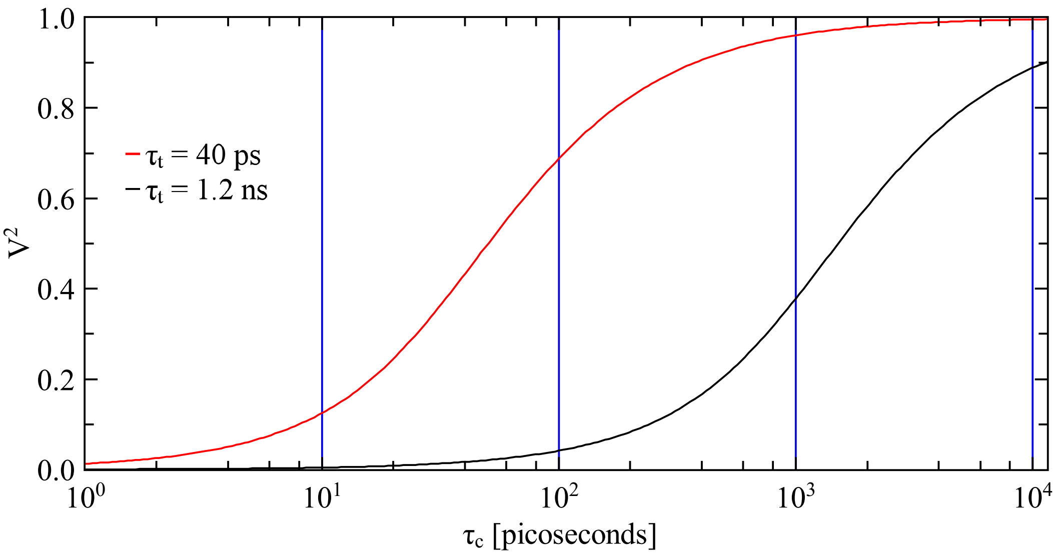

Figure 1: Actual measurable maximal visibility V^2 averaged out over the time bin τ_t that spans $\tau = 0$ to $\tau = \tau_t$, keeping in mind that τ_t is the detector resolution. Note: this measurable visibility is a function of coherence time τ_c, and in fact we observe this effect in measuring the $g^{(2)}$ of the mercury lamp, as to why the peak is 1.8 and not 2, precisely because it is averaged out over the time bin closest to $\tau = 0$.

In the regime of coherence time $\tau_c <<$ detector resolution τ_t, $SNR_{ours} = SNR_{HBT}$
In the regime of coherence time $\tau_c \geq$ detector resolution τ_t, $SNR_{ours} \geq SNR_{HBT}$

8 Generalised SNR

Let us consider the SNR for a single time bin, of width τ_t, and recalling that $I = I_0/\tau_c$, such that the expression for the noise contribution is always

$$Noise = \sqrt{B} = I\sqrt{\tau_t T} = \frac{I_0}{\tau_c}\sqrt{\tau_t T} \tag{24}$$

Let us now consider the signal contribution to be the excess of photon coincidences above the random coincidences, i.e. the area under the visibility curve. Therefore it is perhaps useful to consider an effective visibility $V^{'}$ that is the average of the visibility squared curve from $\tau = 0$ to τ_t, i.e. averaged out within a single time bin and thus the measurable maximal visibility.

$$V^{'} = \frac{1}{\tau_t}\int_{\tau=0}^{\tau=\tau_t} e^{-2\tau/\tau_c} d\tau = \left(\frac{-\tau_c}{2\tau_t} e^{-2\tau/\tau_c}\right)|_{\tau=0}^{\tau=\tau_t} \tag{25}$$

$$V^{'} = \frac{-\tau_c}{2\tau_t} e^{-2\tau_t/\tau_c} + \frac{\tau_c}{2\tau_t} \tag{26}$$

such that the signal contribution is modified (but valid regardless of filtered or not) to:

$$S = I^2 * \tau_t * V^{'} * T \tag{27}$$

For coherence time τ_c much smaller than detector resolution τ_t:

$$V^{'} \approx \frac{\tau_c}{2\tau_t} << 1 \tag{28}$$

simply because the exponential term decays much faster and thus orders of magnitude smaller, leading to

$$S = I^2 * \frac{\tau_c}{2} * T = I_0^2 * \frac{T}{2\tau_c} \tag{29}$$

and thus

$$SNR = \frac{I^2 \frac{\tau_c}{2} T}{I\sqrt{\tau_t T}} = \frac{\tau_c}{2} I \sqrt{\frac{T}{\tau_t}} = \frac{I_0}{2}\sqrt{\frac{T}{\tau_t}} \tag{30}$$

which is independent of spectral filtering as is expected and agrees with referee and conventional wisdom.

Now let us consider the case for coherence time τ_c longer than the detector resolution τ_t such that the in the $V^{'}$ term, the coefficients become larger than 1, and the exponent is smaller than 1:

$$V^{'} = \frac{\tau_c}{2\tau_t}\left(1 - e^{-2\tau_t/\tau_c}\right) \tag{31}$$

such that

$$S_f = I^2 * \tau_t * T * V^{'} = I^2 * \frac{\tau_c}{2} * T * \left(1 - e^{-2\tau_t/\tau_c}\right) \tag{32}$$

and

$$SNR_f = I_0\sqrt{T}\sqrt{\frac{\tau_t}{\tau_c^{2T}}} * V^{'} \tag{33}$$

$$SNR_f = \frac{I_0}{2}\sqrt{\frac{T}{\tau_t}} * \left(1 - e^{-2\tau_t/\tau_c}\right) = SNR * \left(1 - e^{-2\tau_t/\tau_c}\right) \tag{34}$$

9 SNR from the experimental results

$$SNR_f = SNR * \left(1 - e^{-2\tau_t/\tau_c}\right) \tag{35}$$

For $\tau_t = 40\,\text{ps}$ and $\tau_c = 0.3\,\text{ns}$ for the noon solar $g^{(2)}$ measurement:

$$SNR_f = SNR * 0.23 \tag{36}$$

This suggests that the SNR of a single time bin in our measurement, that contains the peak $g^{(2)}$, is 23% that of the SNR of a single time bin that contains all photon bunching signal but without spectral filtering, and holding all other parameters constant.

What is the SNR_f of our single peak time bin then?

Let us look at the noon measurement of the Solar $g^{(2)}(\tau = 0)$. The time bin with the peak value, has 900 coincidence events. The associated visibility or V^2 is 0.37. Thus the number of photon bunching or signal events in this single time bin is $0.37 * 900 = 330$ signal events. The Noise of this bin is $\sqrt{900} = 30$. Thus giving a $SNR = 330/30 = 11$. Does this imply that the uncertainty (or noise) of this time bin is 1/11 or 9% ? i.e. Number of photon coincidences in this bin $= 330 \pm 30$? If this is valid, does it imply $V^2 = 0.37 \pm 9\%$ or 0.37 ± 0.03?

However, if we make use of the information of the other time bins as well, and curve fit in gnuplot, we actually obtained $V^2 = 0.37 \pm 0.01$. The action of fitting thus tripled our statistical confidence in the value of $g^{(2)}(\tau = 0)$ as opposed to just considering a single time bin. Does this imply the effective SNR tripled as well?

10 SNR from papers

Our SNR expression seems reasonable when compared with the referee's, but feels slightly out of place when viewed on its own; specifically, its dependency on the coherence time τ_c is present (strongly so) and a 'wrong' or reversed dependency on the detector resolution τ_t:

$$SNR_{ours} = (I_0)\left(\sqrt{\frac{T}{\tau_t}} \cdot V'\right)\left(\frac{\tau_t}{\tau_c}\right) \tag{37}$$

$$SNR_{HBT} = (A\eta R\, I_0)\left(V^2\sqrt{\frac{T}{2\tau_t}}\right) \tag{38}$$

$$SNR_{iqueye} = \left(\sqrt{N_1 N_2}\,\frac{\tau_c}{2}\right)\left(V^2\sqrt{\frac{T}{2\tau_t}}\right) \tag{39}$$

Adapted from Rou, Nunez, Foellmi, Malvimat who all in turn cited HBT74:

Signal ($\mathrm{photons}^2\,\mathrm{s}^{-2}$) = number of coincidences per unit time bin per unit integration time?

Spectral density I_0 is in units of number of photons, per second of integration, per hertz of optical bandwidth, per metres^2 of aperture area

$$|\gamma(b,\tau)|^2 = V(b,\tau)^2 = \frac{< \delta i_1 \cdot \delta i_2 >}{< i_1 >< i_2 >} \tag{40}$$

b is the spatial baseline separation between the detectors, τ is the timing separation. In our case, $b = 0$ due to the photons being projected into TEM_{00}. Obviously the visibility, and so the signal, and thus the SNR are dependent on the timing separation τ, and is maximal at $\tau = 0$ where $V = 1$.

$$\begin{aligned} g^{(2)}(b,\tau) &= 1 + |\gamma(b,\tau)|^2 && (41)\\ &= \frac{P_{12}}{P_1 \cdot P_2} && (42)\\ &= \frac{I_{12}T}{I_1 T \cdot I_2 T},\ I_1 + I_2 = I = I_0/\tau_c && (43) \end{aligned}$$

But the reality is that $\tau = 0$ cannot be reached, and the closest one can approach is a single time bin that spans $\tau = 0$ to $\tau = \tau_t$, where τ_t is the detector resolution. As such, the maximal visibility, signal and thus SNR are actually averaged out across this time bin with width τ_t.

$$\begin{aligned} Signal &= < \delta i_1 \cdot \delta i_2 > && (44)\\ &= < i_1 >< i_2 > V(\tau)^2 && (45) \end{aligned}$$

where $\delta i_{1,2}$ are the fluctuations in the photodetector currents, and $< \delta i_1 \cdot \delta i_2 >$ is the time-averaged of the product and thus the photon bunching signal. $< i_{1,2} >$ is the time-averaged of the intensities collected by each detector, and thus their uncorrelated product gives the random photon coincidences. The ratio of these two products gives the V^2 or the excess $g^{(2)} - 1$.

To relate from ideal $V^2(\tau)$ to measured $V'(\tau \pm \tau_t/2)$:

$$V^2 = \frac{\tau_c}{\tau_t} V' \tag{46}$$

V^2 is an 'instantaneous' value, and is thus true at all points in time during the measurement duration T. However, V' is a finite measurement that takes τ_t to perform. Therefore it is not a true measure of how 'common' the coincidence events take up the measurement duration T. The scaling term τ_c/τ_t is thus introduced to represent better how 'often' the coincidence events are during the measurement. Is it fair to say detector resolution τ_t constraints the sampling rate of the true coincidences?

So here we show the actual expression for the measured V', which is the true visibility integrated over and then averaged out by the time bin τ_t:

$$
\begin{aligned}
V' &= \frac{1}{\tau_t}\int_{\tau=0}^{\tau=\tau_t} e^{-2\tau/\tau_c} d\tau && (47)\\
&= \left(\frac{-\tau_c}{2\tau_t} e^{-2\tau/\tau_c}\right)|_{\tau=0}^{\tau=\tau_t} && (48)\\
&= \frac{\tau_c}{2\tau_t}\left(1 - e^{-2\tau_t/\tau_c}\right) && (49)
\end{aligned}
$$

$$
\begin{aligned}
Signal &= \left(\frac{A\eta I_0}{\tau_c}\right)_1 \cdot \left(\frac{A\eta I_0}{\tau_c}\right)_2 \cdot \left(\frac{\tau_c}{\tau_t}V'\right) && (50)\\
&= \frac{(A\eta I_0)^2}{\tau_c\tau_t}V' && (51)
\end{aligned}
$$

This Signal final expression is as given by Rou and Nunez citing HBT 74.

Also, keep in mind that the signal here is number of photon coincidences per second of measurement duration per second of time bin; therefore it is not surprising to see that it is independent of measurement duration T, and that the detector resolution τ_t is present only to 'renormalise' the time τ_t taken to measure V'.

$$
\begin{aligned}
random\_events &= Noise^2\_per\_second\_timebin\_per\_second\_duration && (52)\\
&= < i_1 >< i_2 > && (53)\\
&= \left(\frac{A\eta I_0}{\tau_c}\right)^2 \frac{1}{\tau_t T} && (54)
\end{aligned}
$$

Assuming Poissonian noise such that noise is the square-root of the random coincidences rate.

$$
Noise = \frac{A\eta I_0}{\tau_c}\sqrt{\frac{1}{\tau_t T}} \tag{55}
$$

Taking the ratio of these two terms, we arrive at the SNR:

$$
SNR_{ours} = A\eta I_0\sqrt{\frac{T}{\tau_t}} \cdot V' \tag{56}
$$

Assuming an ideal interferometer such that $\tau = 0$ can be measured, then $V' = V^2 = 1$ and we arrive at the SNR_{HBT} in literature which is independent of spectral filtering:

$$
SNR_{HBT} = A\eta I_0\sqrt{\frac{T}{\tau_t}} \cdot V^2 \tag{57}
$$

Assuming a real interferometer such that the closest to $\tau = 0$ is an averaged measurement between $\tau = 0$ to $\tau = \tau_t$, then SNR is weakly (effectively not) dependent on spectral filtering as V' is flat, until the coherence time τ_c approaches the detector resolution τ_t:

$$SNR_{ours} = A\eta I_0\sqrt{\frac{T}{\tau_t}} \cdot \frac{\tau_c}{2\tau_t}\left(1 - e^{-2\tau_t/\tau_c}\right) \tag{58}$$

In the regime of $\tau_c < \tau_t$, $SNR_{ours} = SNR_{HBT}$.
In the regime of $\tau_c \geq \tau_t$, $SNR_{ours} \geq SNR_{HBT}$