The Hanbury Brown-Twiss Intensity Interferometer

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Abstract

The semi-classical two-correlation function for the Hanbury Brown-Twiss Intensity Interferometer is derived.

Chapter 1

Theory

1.1 The two-correlation function, C

The two-correlation function for two identical particles detected within a short time period as a function of the momentum difference of the particles, $|\vec{q}| = |\vec{p_1} - \vec{p_2}|$, is:

$$C\left(|\vec{q}|\right) = \frac{\{\langle n_{\vec{p_1}} n_{\vec{p_2}} \rangle\}}{\{\langle n_{\vec{p_1}} \rangle \langle n_{\vec{p_2}} \rangle\}}, \tag{1.1}$$

where $n_{\vec{p_i}}$ is the number of particles of momentum $\vec{p_i}$ observed in a single event, the angle brackets, $\langle \cdots \rangle$, denote an average over a number of events and the curled brackets, $\{\cdots\}$, denote an average over a range of centre of mass momenta of the particle pair, $\vec{P} = \vec{p_1} + \vec{p_2}$.

1.2 The Intensity at a Dectector

We shall consider one photon, γ, emitted from each source. The photon emitted from source a, γ_a, is described by:

$$|\gamma_a\rangle = \frac{\alpha}{|\vec{r} - \vec{r_a}|} e^{ik|\vec{r} - \vec{r_a}| + i\phi_a}. \tag{1.2}$$

Similarly, the photon emitted from source b, γ_b, is described by:

$$|\gamma_b\rangle = \frac{\beta}{|\vec{r} - \vec{r_b}|} e^{ik|\vec{r} - \vec{r_b}| + i\phi_b}, \tag{1.3}$$

where $\phi_{a,b}$ are random phases.

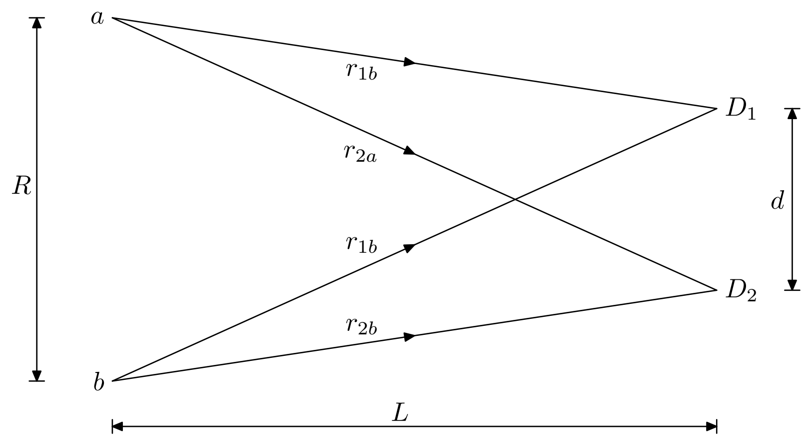
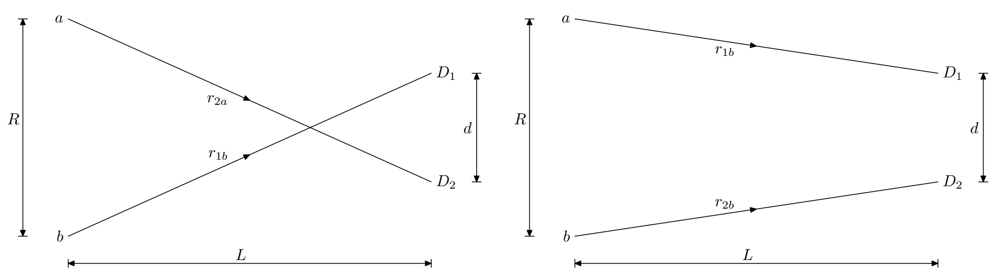


Figure 1.1: Schematic diagram of the Hanbury Brown-Twiss Intensity Interferometer. The arrival times of two identical particles are observed at two detectors, after being emitted at seperate points of a source located at a distance, L. The emission points, a and b, are seperated by a distance, R. The detectors, D_1 and D_2, are seperated by a distance, d.



(a) Figure a

(b) Figure b

Figure 1.2: The two indistinguishable paths the particles may take.

The total amplitude at detector D_1, A_1 is given by:

$$A_1 = \frac{1}{L}\left(\alpha e^{ikr_{1a}+i\phi_a} + \beta e^{ikr_{1b}+i\phi_b}\right), \quad (1.4)$$

where r_{1a} denotes the distance from source a to detector D_1, and so on. To calculate the intensity at a detector, we calculate the square of the absolute value of the amplitude. For example, at detector D_1:

$$\begin{aligned} I_1 &= A_1^* A_1 \\ &= \frac{1}{L^2}\left(|\alpha|^2 + |\beta|^2 + \alpha^*\beta e^{ik(r_{1b}-r_{1a})+i\phi_b-i\phi_a} + \alpha\beta^* e^{-ik(r_{1b}-r_{1a})-i\phi_b+i\phi_a}\right). \end{aligned} \tag{1.5}$$

Averaging over the random phases, denoted by terms in angled brackets like so: $\langle\cdots\rangle_{\phi_i}$, we can see that the exponential terms average to zero and equation 1.5 simplifies to:

$$\langle I_1\rangle_{\phi_i} = \frac{1}{L^2}\left(|\alpha|^2 + |\beta|^2\right). \tag{1.6}$$

1.3 The Simultaneous Intensity at Both Detectors

To calculate the product of the intensities, first we define:

$$\phi_{ba} = \phi_b - \phi_a, \tag{1.7}$$

$$R_{1ba} = r_{1b} - r_{1a}. \tag{1.8}$$

Using this notation the intensity is now given by:

$$I_1 = \frac{1}{L^2}\left(|\alpha|^2 + |\beta|^2 + \alpha^*\beta e^{ikR_{1ba}+i\phi_{ba}} + \alpha\beta^* e^{-ikR_{1ba}-i\phi_{ba}}\right) \tag{1.9}$$

We may now expand $I_1 I_2$ in this, more compact, form:

$$\begin{aligned} I_1 I_2 &= \frac{1}{L^2}\left(|\alpha|^2 + |\beta|^2 + \alpha^*\beta e^{ikR_{1ba}+i\phi_{ba}} + \alpha\beta^* e^{-ikR_{1ba}-i\phi_{ba}}\right) \\ &\quad \times \frac{1}{L^2}\left(|\alpha|^2 + |\beta|^2 + \alpha^*\beta e^{ikR_{2ba}+i\phi_{ba}} + \alpha\beta^* e^{-ikR_{2ba}-i\phi_{ba}}\right) \end{aligned} \tag{1.10}$$

expanding and collecting terms we arrive at:

$$\begin{aligned} I_1 I_2 \;&=\; \frac{1}{L^4}\big(|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 \\ &+2e^{ikR_{2ba}+i\phi_{ba}}\left[|\alpha|^2\alpha^*\beta + |\beta|^2\alpha\beta^*\right] \\ &+2e^{-ikR_{2ba}-i\phi_{ba}}\left[|\alpha|^2\alpha\beta^* + |\beta|^2\alpha^*\beta\right] \\ &+e^{ikR_{1ba}+i\phi_{ba}+ikR_{2ba}+i\phi_{ba}}\left[\alpha^*\beta\alpha^*\beta\right] \\ &+e^{ikR_{1ba}+i\phi_{ba}-ikR_{2ba}-i\phi_{ba}}\left[|\alpha|^2|\beta|^2\right] \\ &+e^{-ikR_{1ba}-i\phi_{ba}+ikR_{2ba}+i\phi_{ba}}\left[|\alpha|^2|\beta|^2\right] \\ &+e^{-ikR_{1ba}-i\phi_{ba}-ikR_{2ba}-i\phi_{ba}}\left[(\alpha\beta^*)^2\right]\big). \end{aligned} \tag{1.11}$$

We can further simplify the expression:

$$\begin{aligned} I_1 I_2 \;&=\; \frac{1}{L^4}\big(|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 \\ &+2e^{ikR_{2ba}+i\phi_{ba}}\left[|\alpha|^2\alpha^*\beta + |\beta|^2\alpha\beta^*\right] \\ &+2e^{-ikR_{2ba}-i\phi_{ba}}\left[|\alpha|^2\alpha\beta^* + |\beta|^2\alpha^*\beta\right] \\ &+e^{ikR_{1ba}+ikR_{2ba}+2i\phi_{ba}}\left[\alpha^*\beta\alpha^*\beta\right] \\ &+e^{ikR_{1ba}-ikR_{2ba}}\left[|\alpha|^2|\beta|^2\right] \\ &+e^{-ikR_{1ba}+ikR_{2ba}}\left[|\alpha|^2|\beta|^2\right] \\ &+e^{-ikR_{1ba}-ikR_{2ba}-2i\phi_{ba}}\left[(\alpha\beta^*)^2\right]\big). \end{aligned} \tag{1.12}$$

Again, if we average over the random phases, we see that all exponential terms containing the phases ϕ_i average to zero and we are left with:

$$\begin{aligned} \langle I_1 I_2\rangle_{\phi_i} \;&=\; \frac{1}{L^4}\big(|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 \\ &+|\alpha|^2|\beta|^2\left[e^{ikR_{1ba}-ikR_{2ba}} + e^{-ikR_{1ba}+ikR_{2ba}}\right]\big). \end{aligned}$$

The last term in Eq. 1.13 can be re-written as a cosine:

$$\begin{aligned} \langle I_1 I_2\rangle_{\phi_i} \;&=\; \frac{1}{L^4}\big(|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 \\ &+2|\alpha|^2|\beta|^2\cos\left(ikR_{1ba} - ikR_{2ba}\right)\big), \end{aligned}$$

or, in the original notation:

$$\begin{aligned} \langle I_1 I_2\rangle_{\phi_i} \;&=\; \frac{1}{L^4}\big(|\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 \\ &+2|\alpha|^2|\beta|^2\cos\left(ik\left(r_{1b} - r_{1a} - r_{2b} + r_{2a}\right)\right)\big). \end{aligned} \tag{1.13}$$

We note that the first three terms are exactly equal to the product of the random phase-averaged intensities:

$$\langle I_1 I_2 \rangle_{\phi_i} = \frac{1}{L^4}\Big(\langle I_1\rangle_{\phi_i}\langle I_2\rangle_{\phi_i} + 2\left|\alpha\right|^2\left|\beta\right|^2\cos\left(ik\left(r_{1b}-r_{1a}-r_{2b}+r_{2a}\right)\right)\Big). \tag{1.14}$$

The two-particle correlation function for one pair of plane waves is then:

$$\begin{aligned} C_{\phi_i}\left(\vec{d}\right) &= \frac{\langle I_1 I_2\rangle_{\phi_i}}{\langle I_1\rangle_{\phi_i}\langle I_2\rangle_{\phi_i}} \\ &= \frac{\langle I_1\rangle_{\phi_i}\langle I_2\rangle_{\phi_i}}{\langle I_1\rangle_{\phi_i}\langle I_2\rangle_{\phi_i}} + \frac{2\left|\alpha\right|^2\left|\beta\right|^2\cos\left(ik\left(r_{1b}-r_{1a}-r_{2b}+r_{2a}\right)\right)}{\langle I_1\rangle_{\phi_i}\langle I_2\rangle_{\phi_i}} \\ &= 1+\frac{2\left|\alpha\right|^2\left|\beta\right|^2\cos\left(ik\left(r_{1b}-r_{1a}-r_{2b}+r_{2a}\right)\right)}{\langle I_1\rangle_{\phi_i}\langle I_2\rangle_{\phi_i}} \\ &= 1+\frac{2\left|\alpha\right|^2\left|\beta\right|^2}{\left(\left|\alpha\right|^2+\left|\beta\right|^2\right)^2}\cos\left(ik\left(r_{1b}-r_{1a}-r_{2b}+r_{2a}\right)\right) \end{aligned} \tag{1.15}$$

The expression $(r_{1b}-r_{1a}-r_{2b}-r_{2a})$ represents the difference in phase difference between the two paths in figure 1.2. If the detectors are well-seperated from the sources, $L \gg R$, then:

$$\begin{aligned} k\left(r_{1b}-r_{1a}-r_{2b}+r_{2a}\right) &\rightarrow k\left(r_b-r_a\right)\cdot\left(r_2-r_2\right) && (1.16) \\ &= \vec{R}\cdot\left(\vec{k_2}-\vec{k_1}\right) && (1.17) \end{aligned}$$

Possibly more intuitively, we can calculate the difference in phase difference using the diagram presented in figure 1.3. The difference in phase difference, $k\left(d_2-d_1\right)$, can be expressed as:

$$k\left(d_2-d_1\right) = \frac{2\pi}{\lambda} d\sin\left(\theta\right). \tag{1.18}$$

Using the small angle approximation, $\sin\left(\theta\right)\rightarrow\theta$ and $\theta\rightarrow R/L$, and remembering that the wavevector, k, is equal to $2\pi/\lambda$, this gives:

$$d_2-d_1 = \frac{2\pi}{\lambda} d\cdot\theta \tag{1.19}$$

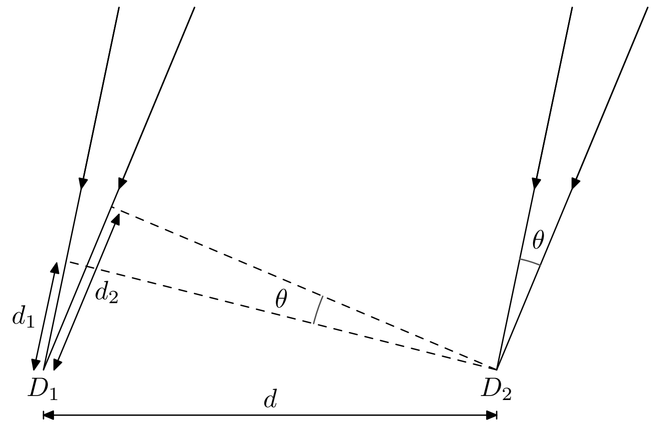


Figure 1.3: The difference in phase difference shown diagrammatically. Using the small angle approximation, we can see that $d_2 - d_1$ is given by $d\theta$.

Inserting 1.19 into 1.15 we can see that the two-correlation function varies as a function of both the separation of the detectors but also of the angular diameter of the source:

$$C\left(\left|\vec{d}\right|\right) = 1 + \frac{2\left|\alpha\right|^2\left|\beta\right|^2}{\left(\left|\alpha\right|^2 + \left|\beta\right|^2\right)^2}\cos\left(\frac{2\pi d\theta}{\lambda}\right) \tag{1.20}$$

Of course, this is not a full treatment of the phenomena and we have made a number of simplifications in this semi-classical treatment. Namely: the polisarisation of the photons was completely neglected, the radiating atoms never decay, the possibility of different arrival times was not considered, ...

1.4 Caveat

A full derivation can found in [1]. The final two-correlation, given by summing over all possible pairs of atoms, polarisations, etc. is of the form:

$$\sum_{12}\mathcal{P}_{12}\left(t\right) = \int_0^t dt_1 \int_0^t dt_2 p\left(t_1, t_2\right), \tag{1.21}$$

where, t_1, t_2, are the times at which photoionisation occurs in an atom in detector 1 and 2 respectively, and $p\left(t_1, t_2\right)$ is a doubly-differential probability

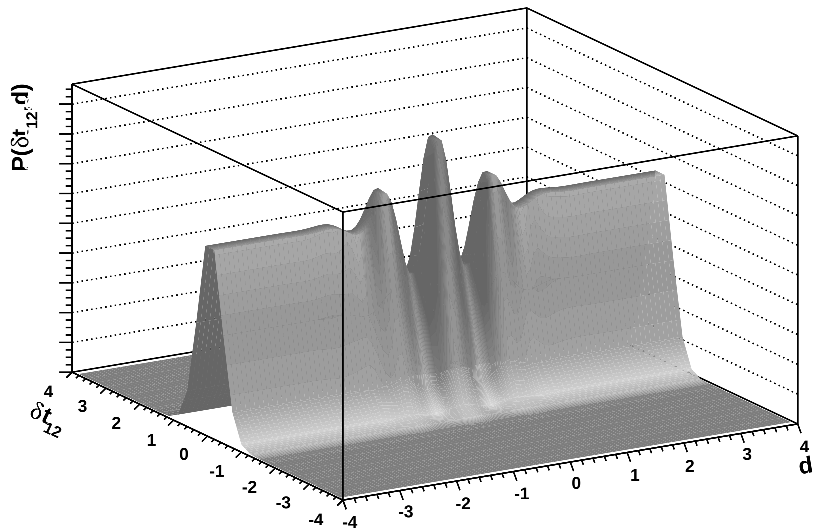


Figure 1.4: The basic shape of the doubly-differential probability distribution as a function of the emission time difference, δt_{12}, and detector separation, d.

distribution described to lowest non-vanising order by:

$$\begin{aligned} p\left(t_1, t_2\right) \quad \propto \quad & e^{-2\Gamma_{ab}\left(t_1+t_2-2R\right)} \times \\ & \left\{A \cosh\left[\Delta\Gamma_{ab}\left(t_1-t_2\right)\right] + B\cos\left[k\left(r_{1a}-r_{1b}-r_{2a}+r_{2b}\right)-\Delta E_{ab}\left(t_1-t_2\right)\right]\right\} \times \\ & \mathrm{St}\left(t_1\right)\mathrm{St}\left(t_2\right), \end{aligned} \tag{1.22}$$

where $\Gamma_{a,b}$ are the decay constants, $E_{a,b}$ are are excitation energies of the excited stated of the radiating atoms, $\phi_{a,b}$ again represent the phase constants, and $\mathrm{St}\left(x\right)$ is a step function representing the extent of the emitted wave. The basic shape of the probability function is shown in figure 1.4.

References

[1] U. Fano, American Journal of Physics **29**, 539 (1961).

[2] G. Baym, Acta Phys. Polon. **B29**, 1839 (1998), nucl-th/9804026.