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Abstract. We present a Lorentz invariant extension of a previous model for intrinsic decoherence (Milburn 1991 *Phys. Rev.* A **44** 5401). The extension uses unital semigroup representations of space and time translations rather than the more usual unitary representation, and does the least violence to physically important invariance principles. Physical consequences include a modification of the uncertainty principle and a modification of field dispersion relations, similar to modifications suggested by quantum gravity and string theory, but without sacrificing Lorentz invariance. Some observational signatures are discussed.

Contents

1. Introduction

The precision with which intervals of time and length can be measured is limited by intrinsic quantum uncertainties [1]. The limit on precision is determined by fundamental constraints on estimating the parameter of an appropriate time or space translation. These limits arise from the statistical distinguishability of quantum states and reflect the geometry of Hilbert space itself.

In practice, however, precision is limited by interactions between the measured system and other degrees of freedom (the environment) over which we have little control. Such interactions add noise to the measurement outcomes, reflecting our lack of knowledge of the precise state of the environment. Complementary to added noise is decoherence: the process through which interactions with the environment destroy coherence between superposed quantum states in some specific basis. Studies of environmentally induced decoherence over the last three decades have given a reasonably good picture of the process [2]–[4], although detailed comparison to experiment is relatively recent.

Quantum decoherence can also arise due to classical fluctuations in the parameters which define the dynamics of the system. In this case, decoherence is found when data from repeated trials are combined without regard to the fluctuations in the parameters defining the experimental conditions from trial to trial. For example, in a 'Ramsey fringe' experiment [5], the probability to find a two-level system in a particular state is sampled over many trials in which the time between state preparation and measurement is supposedly fixed. However, fluctuations in this time interval will appear as a dephasing of the Rabi oscillations between the two levels involved. In a more fundamental setting, fluctuations in the space-time metric would correspond to a source of intrinsic noise, which would necessarily be accompanied by intrinsic decoherence [6].

Decoherence is often invoked to explain the lack of quantum effects in macroscopic systems. While this is often the case for environment induced decoherence, a number of authors [7]–[12] have speculated that an intrinsic decoherence may exist to establish classical behaviour at some level. In this paper, we extend a previous approach based on stochastic time [9]. This provides a path to a Lorentz invariant field theoretical formulation of a model of intrinsic decoherence.

These heuristic modifications of the Schrödinger equation should more properly be viewed in a like manner to environmental decoherence, but in which the quantum nature of the environment is left unspecified. Recently Gambini *et al* [13] have shown that a recent proposal for quantization of gravity, based on discrete space-time, is consistent with the model of intrinsic decoherence discussed in [9]. The extension proposed in this paper likewise has consequences that have previously been considered in the context of quantum gravity. Intrinsic decoherence due to spatial displacements leads to a modification of the uncertainty principle which is similar to that considered in the context of quantum gravity [14]–[16]. When extended to the relativistic case, in particular the electromagnetic field, we find that the dispersion relation for the free field must be modified. A similar effect has also been suggested for models of quantum space-time [17].

Space and time parameterize fundamental symmetry groups. The action of a group element on a physical state is represented by a unitary operator on Hilbert space. Conventionally, we consider continuous representations of these symmetries which reflect the strong classical intuition that space and time are continuous parameters. We can define the unitary representations through their infinitesimal action. The unitary representation is then defined in terms of a hermitian operator which is the generator of the group. In the case of time translations, the generator is the Hamiltonian operator, while in the case of spatial translation the generator is the momentum operator. In a relativistic theory, these operators are constructed from the quantum fields that define the physical systems under investigation.

In non-relativistic quantum mechanics, spatial and temporal translations are represented by the one-parameter unitary transformations,

$$\rho(X) = \mathrm{e}^{-\mathrm{i}X\hat{p}/\hbar}\rho_0\mathrm{e}^{\mathrm{i}X\hat{p}/\hbar}, \tag{1}$$

$$\rho(t) = \mathrm{e}^{-\mathrm{i}t\hat{H}/\hbar}\rho_0\mathrm{e}^{\mathrm{i}t\hat{H}/\hbar}, \tag{2}$$

or in differential form,

$$\frac{\mathrm{d}\rho(X)}{\mathrm{d}X} = -\frac{\mathrm{i}}{\hbar}[\hat{p}, \rho(X)], \tag{3}$$

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[\hat{H}, \rho(t)]. \tag{4}$$

The representation of the group of spatial and temporal translations in terms of the unitary operators generated by energy and momentum of course leads directly to the conservation of energy and momentum. How can we modify these rules causing the least violence to energy and momentum conservation? To answer this, we need to consider in a little more detail the kinds of experiments that enable us to estimate space and time translations. The high precision measurement of space and time intervals are in fact determinations of the statistical distinguishably of quantum states through repeated preparation and measurement.

2. Estimating space-time translations

To make a quantum clock, we must superpose at least two distinct energy eigenstates. Indeed this is exactly how time is currently measured with atomic clocks [18]. Let the system be prepared, at time $t = 0$, in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle), \tag{5}$$

where $|E_i\rangle$ are energy eigenstates. The variance of the energy in this state is $\langle\Delta\hat{H}^2\rangle_0 = \Delta_E^2/4$ with $\Delta_E = E_2 - E_1$.

After a time t, in accordance with unitary temporal displacement, the state becomes

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\mathrm{e}^{-\mathrm{i}\omega_1 t}|E_1\rangle + \mathrm{e}^{-\mathrm{i}\omega_2 t}|E_2\rangle), \tag{6}$$

where $\omega_i = E_i/\hbar$.

The next step is to measure some quantity represented by an operator that does not commute with the Hamiltonian. The simplest choice is the projection operator $\hat{P}_+ = |+\rangle\langle+|$ on to the state

$|+\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle)$. There are two possible values, $x = 0, 1$ for the measurement result, with probability distribution

$$p_1(t) = 1 - p_0(t) = \cos^2\left(\frac{\Delta_E t}{2\hbar}\right). \tag{7}$$

There are two ways, this system may be used as a clock. Both cases require us to sample the probability distribution in equation (7) and thus both require that we prepare a large number of identical systems in the manner just described and measure the quantity $\hat{P}_+$ on each of them.

The first and most direct method is simply to measure the quantity $\hat{P}_+$ to sample the probability $p_x(t)$ and thus infer t. Of course this inference must come with some error which can easily be determined. The second way is to note that the parameter $\Delta_E/\hbar$ is a frequency. We can tune an external signal, such as a laser, to this frequency and then use the sampling of the ensemble of systems to keep the signal frequency locked on a particular value using feedback control.

It is a simple matter to estimate the uncertainty with which we can infer the parameter t. It suffices to measure the quantity $\hat{P}_+ = |+\rangle\langle+|$ on the state given in equation (6). The average value of this quantity is the probability $p_1(t)$. The uncertainty in this measurement is

$$\Delta p_1 = \sqrt{p_1(1 - p_1)}. \tag{8}$$

The uncertainty in the inference of the time parameter t, is then given by [19]

$$\delta t = \left|\frac{\mathrm{d}p_1(t)}{\mathrm{d}t}\right|^{-1} \Delta p_1. \tag{9}$$

Thus we find the well-known result

$$\delta t = \frac{\hbar}{|\Delta_E|} \tag{10}$$

Noting that the variance of the energy for the fiducial state is $\langle \Delta \hat{H}^2\rangle_0 = \Delta_E^2/4$, we see that the quality of the inference varies as the inverse of the energy uncertainty. This is the standard result for a parameter based uncertainty principle [1].

To summarize, time measurement in a quantum world requires us to sample a probability distribution by making measurements on an ensemble of identically prepared systems. A single system or a single measurement would not do.

In complete analogy with the estimation of time intervals, we can make a 'quantum ruler' for the estimation of spatial intervals by using a superposition of momentum eigenstates of equal and opposite momentum for a free particle,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|p_1\rangle + |p_2\rangle). \tag{11}$$

This leads to a standing wave for the position probability amplitude. The state is an energy eigenstate of a free particle, and the resulting standing wave will not change in time. However the state is not an eigenstate of momentum; the momentum uncertainty is given by $\langle \hat{p}^2 \rangle_0 = \Delta_p^2/4$ where $\Delta_p = p_2 - p_1$.

In writing this state with the particular choice of relative phase between the two components, we are assuming that the origin is located at an antinode of the standing wave. If we translate the ruler to a new position, labelled X, the state changes in accordance with the unitary representation of spatial translations as

$$|\psi(X)\rangle = \mathrm{e}^{-\mathrm{i}k_1 X}|p_1\rangle + \mathrm{e}^{-\mathrm{i}k_2 X}|p_2\rangle, \tag{12}$$

where $k = p/\hbar$. In analogy with the time example of the previous section, we choose to measure the operator conjugate to momentum, i.e. we measure position. The probability distribution to get a particular result, x, is then

$$P(x) \propto \cos^2\left(\frac{\Delta_p(X-x)}{2\hbar}\right). \tag{13}$$

To estimate the translation parameter, we need to sample this distribution. Of course, the determination of X will be accompanied by some necessary error of a quantum origin. This is roughly the distance between two successive minima of the probability density, $P(x)$. In this case, one easily sees that this is a consequence of the Heisenberg uncertainty principle resulting from the initial momentum uncertainty of the state in equation (14) [1]. The uncertainty in the inferred value of X is given by

$$\delta X \geqslant \frac{\hbar}{|\Delta_p|}. \tag{14}$$

In these two examples, we see that a determination of temporal duration and space translation require to sample a probability distribution. This leads to the well known intrinsic quantum uncertainty limits for time and position parameter estimation [1]. The process of preparing the ensemble (either by trying to prepare a large number of identical systems, or a process of preparation and re-preparation of a single system) is in practice subject to additional sources of error. Furthermore, the measurements are not always perfect and noise may be added from trial to trial. For this reason, the actual state used to describe an ensemble may not necessarily be simply a product of identical pure states, but may rather be a density operator reflecting some additional degree of averaging over unknown sources of noise and error. The fact that space and time parameters must be inferred by sampling a probability distribution over an ensemble is an important insight and suggests a path for developing a theory of intrinsic decoherence.

3. Intrinsic decoherence

We now modify the unitary representation of space and time translation by using semigroup representations. We first recall that, in estimating a parameter, multiple trials must be performed and in practice it may not be possible to ensure that each trial is identical. Suppose now that *in principle* it is impossible for each trial to be identical, for some fundamental reason. In that case parameter estimation would necessarily be based on a mixed state rather than a pure state. To be more specific we suppose that for each unitary representation of the parameter transformation there is a minimum Hilbert space rotation angle, ϵ and further that the *number* of such rotations, from one trial to the next, can fluctuate. Let $p_n(\theta, \epsilon)$ be the probability that there are n such phase shifts for a change in the macroscopic parameter from 0 to θ. We obtain different models for each choice of the probability function $p_n(\theta, \epsilon)$. Thus the state $\rho(\theta)$ may be written

$$\rho_\epsilon(\theta) = \sum_{n=0}^{\infty} p_n(\theta, \epsilon) \mathrm{e}^{-\mathrm{i}n\epsilon\hat{g}/\hbar} \rho(0) \mathrm{e}^{\mathrm{i}n\epsilon\hat{g}/\hbar}, \tag{15}$$

where $\hat{g} \to \hat{p}$ for spatial translations and $\epsilon \to \mu$ with units of length, while $\hat{g} \to \hat{H}$ for temporal translations, and $\epsilon \to \nu$ with units of time. These assumptions are equivalent to assuming that space and time are discrete with fundamental scales determined by ν, μ. A recent formulation of quantum gravity, using a relational approach, by Gambini *et al* [13] gave a very similar result to that in equation (15).

We also require that, in some limit, the standard unitary representation is obtained. To this end we require

$$\lim_{\epsilon \to 0} \rho_\epsilon(\theta) = \rho(\theta) = \mathrm{e}^{-\mathrm{i}\hat{g}\theta/\hbar} \rho(0) \mathrm{e}^{\mathrm{i}\hat{g}\theta/\hbar}. \tag{16}$$

This condition imposes a restriction on the permissible forms of $p_n(\theta, \epsilon)$.

It is easiest to define the semigroup in terms of its infinitesimal generator. There is a great deal of freedom in how we do this corresponding to different choices for the probability distribution for the number of phase shifts. However, we stipulate that it must respect the conservation of energy and momentum. We will use the differential form of the parameter transformation

$$\frac{\mathrm{d}\rho(\theta)}{\mathrm{d}\theta} = \mathcal{D}[\hat{G}]\rho(\theta), \tag{17}$$

where θ is the parameter, $\mathcal{D}[\hat{G}]$ is the generator of a completely positive semigroup map defined by $\mathcal{D}[\hat{G}]\rho = \hat{G}\rho\hat{G}^\dagger - \frac{1}{2}[\hat{G}^\dagger\hat{G}\rho + \rho\hat{G}^\dagger\hat{G}]$. We require that for spatial translations, with generator $\hat{S}$, $\mathcal{D}[\hat{S}]\hat{p} = 0$, while for temporal translations, with generator $\hat{T}$, $\mathcal{D}[\hat{T}]\hat{H} = 0$. This ensures that momentum and energy are conserved in the semigroup transformation. As a specific example, we will take each generator to be a unitary operator of the form;

$$\hat{G} = \epsilon^{-1/2} \mathrm{e}^{-\mathrm{i}\hat{g}\epsilon/\hbar}, \tag{18}$$

which we shall refer to as the *unital* case. Substituting equation (18) into equation (17) indicates that

$$\frac{\mathrm{d}p_n(\theta, \epsilon)}{\mathrm{d}\theta} = \frac{1}{\epsilon}(p_{n-1}(\theta, \epsilon) - p_n(\theta, \epsilon)), \tag{19}$$

$$\frac{\mathrm{d}p_0(\theta, \epsilon)}{\mathrm{d}\theta} = -\frac{1}{\epsilon} p_0(\theta, \epsilon). \tag{20}$$

The solution is

$$p_n(\theta, \epsilon) = \frac{(\theta/\epsilon)^n}{n!} e^{-\theta/\epsilon}. \tag{21}$$

For obvious reasons we call this the Poisson choice. In the limit that $\epsilon \to 0$, we recover the standard Schrödinger representation from equation (17),

$$\frac{d\rho(\theta)}{d\theta} = -\frac{i}{\hbar}[\hat{g}, \rho(\theta)]. \tag{22}$$

The condition in equation (16) is thus satisfied. We anticipate that ν, μ ultimately take their values from a future quantum theory of space-time, so here we simply equate them to the Planck time and Planck length, respectively, in which case $c\nu = \mu$. We now consider the experimental consequences of this modification of the Schrödinger rules.

We begin with temporal translations. The differential change in a state due to a temporal translation, t, is given by [9]

$$\frac{d\rho(t)}{dt} = \frac{1}{\nu}\left[e^{-i\nu\hat{H}/\hbar}\rho(t)e^{i\nu\hat{H}/\hbar} - \rho(t)\right], \tag{23}$$

where ν is a fundamental constant with units of time (in terms of reference [9] $\gamma = 1/\nu$). The physical consequences of this equation have been explored in [9] and subsequent papers [20]. Firstly, the standard limit (equation (10)) for the uncertainty in the estimate of a time parameter is changed to include an additional noise source. Secondly, and most relevant for this paper, the dynamics implied by equation (23) lead to the decay of coherence in the energy basis.

We will use the example discussed in section 10. The time parameter uncertainty bound can be conveniently written in terms of the average value of the Hermitian operator $\hat{X} = |E_1\rangle\langle E_2| + |E_2\rangle\langle E_1|$ as

$$\sqrt{1 - \langle\hat{X}(t)\rangle^2}\left|\frac{d\langle\hat{X}(t)\rangle}{dt}\right|^{-1}. \tag{24}$$

The equation of motion for $\langle\hat{X}(t)\rangle$ can then be found using equation (23). The solution is [9]

$$\langle\hat{X}(t)\rangle = \Re\left\{\exp\left[-\frac{t}{\nu}(1 - e^{-i\nu\omega})\right]\right\}, \tag{25}$$

where $\omega = \Delta_E/\hbar$. The resulting bound on δt is complicated, but can be simplified by the case $\nu\omega \ll 1$, for which we can approximate

$$\langle\hat{X}(t)\rangle \approx e^{-\nu\omega^2 t/2}\cos(\omega t). \tag{26}$$

We expect that the best accuracy for the estimate of time will occur when $\langle\hat{X}(t)\rangle = 0$, as at that point this moment has maximum slope. This corresponds to the condition $\cos(\omega t) = 0$. At those times we find

$$\delta t \approx \frac{1}{\Delta_E} e^{\nu\omega^2 t/2}. \tag{27}$$

For short times, $\Delta t \ll 1$ this agrees with the standard time parameter uncertainty bound for this system. However, for long times we see that there is an exponential degradation of the accuracy. This result suggests that clocks 'age', that is to say long-lived atomic clocks gradually lose stability. This is what one would expect for a clock transition with an intrinsic dephasing rate of $\delta\omega = \nu\omega^2/2$ [21].

The consequences for estimating a temporal translation are important as they indicate a fundamental limitation on the accuracy of clocks. This is an aspect that has been considered in some detail by Egusquiza and Garay [22], from a very different starting point.

Now consider the case of spatial translations. Suppose we take as the fiducial state a system in a pure, minimum uncertainty, state, $|\psi_0\rangle$ with a Gaussian position probability density;

$$P_0(x) = |\langle\psi_0|x\rangle|^2 = (2\pi\sigma)^{-1/2}\mathrm{e}^{-x^2/2\sigma}, \tag{28}$$

where

$$\sigma = \langle\Delta\hat{x}^2\rangle_0 = \frac{\hbar^2}{2\langle\Delta\hat{p}^2\rangle_0}, \tag{29}$$

where $\langle\Delta\hat{A}^2\rangle_0$ is the variance of the operator $\hat{A}$ in the fiducial state. Under the conventional Schrödinger rule for displacements this fiducial density conditioned on a displacement, X, becomes

$$P^{(c)}(x|X) = (2\pi\sigma)^{-1/2}\mathrm{e}^{-(x-X)^2/2\sigma}. \tag{30}$$

We can see that the uncertainty, δX with which we can infer the parameter, X is $\delta X \geqslant \sqrt{\sigma}/2$ or in other words $\delta X^2\langle\Delta\hat{p}^2\rangle_0 \geqslant \hbar^2/4$, which is the standard result for a parameter-based uncertainty principle for position [1].

In the modified Schrödinger rule, this uncertainty principle is modified as the width of the position distribution is no longer independent of the displacement but increases linearly with displacement. The change in the state due to the displacement is given by,

$$\frac{\mathrm{d}\rho(X)}{\mathrm{d}X} = \frac{1}{\mu}\left(\mathrm{e}^{-\mathrm{i}\mu\hat{p}/\hbar}\rho(X)\mathrm{e}^{\mathrm{i}\mu\hat{p}/\hbar} - 1\right). \tag{31}$$

This equation appears to bear a superficial relation to the recent proposal of Shalyt-Margolin and Suarez [23].

To see how the uncertainty principle is changed, we use equation (31) to find an equation for rate of change of the mean position and variance with displacement. It is easy to see that

$$\frac{\mathrm{d}\langle\hat{x}\rangle}{\mathrm{d}X} = 1, \qquad \frac{\mathrm{d}\langle\hat{x}^2\rangle}{\mathrm{d}X} = 2\langle\hat{x}\rangle + \mu.$$

Thus for the chosen fiducial state,

$$\langle\Delta\hat{x}^2\rangle_X = \langle\Delta\hat{x}^2\rangle_0 + \mu X. \tag{32}$$

The uncertainty with which we can estimate the parameter now becomes,

$$\delta X^2 \geqslant \langle \Delta \hat{x}^2 \rangle_0 + X\mu, \tag{33}$$

which implies the uncertainty principle

$$\delta X^2 \langle \Delta \hat{p}^2 \rangle_0 \geqslant \frac{\hbar^2}{4} + X\mu \langle \Delta \hat{p}^2 \rangle_0. \tag{34}$$

This kind of modified uncertainty principle has been suggested in the context of quantum gravity and string theory [15]. We see here that it arises as a natural consequence of an intrinsic uncertainty of spatial translations.

We turn from intrinsic noise to the complementary process of intrinsic decoherence. In quantum mechanics noise is necessarily accompanied by decoherence. Thus, any model that introduces an intrinsic uncertainty due to space-time fluctuations must necessarily introduce intrinsic decoherence. In the case of temporal translations, the decoherence occurs in the energy basis as is easily seen by computing the change in the off-diagonal elements of the state in the energy basis. From equation (23) we see that

$$\frac{\mathrm{d}\rho_{i,j}(t)}{\mathrm{d}t} = \frac{1}{\nu} \left(\mathrm{e}^{-\mathrm{i}\nu(E_i - E_j)/\hbar} - 1 \right) \rho_{i,j}(t), \tag{35}$$

where $\rho_{i,j}(t) = \langle E_i | \rho(t) | E_j \rangle$ with $|E_i\rangle$ an energy eigenstate. This equation was discussed extensively in [9]. To see the effect of intrinsic decoherence we expand the right-hand side to first order in ν,

$$\frac{\mathrm{d}\rho_{i,j}(t)}{\mathrm{d}t} = -\mathrm{i}\frac{(E_i - E_j)}{\hbar}\rho_{i,j}(t) - \frac{\nu(E_i - E_j)^2}{2\hbar^2}\rho_{i,j}(t). \tag{36}$$

The last term induces a decay of off-diagonal matrix elements in the energy basis at a rate that increases quadratically with distance away from the diagonal.

In the case of spatial translations, we find a similar equation that causes a decay with respect to the translation parameter of off-diagonal matrix elements in the momentum basis,

$$\frac{\mathrm{d}\rho_{k,k'}(t)}{\mathrm{d}X} = \frac{1}{\mu} \left(\mathrm{e}^{-\mathrm{i}\mu(k-k')} - 1 \right) \rho_{k,k'}(t), \tag{37}$$

where $\rho_{k,k'}(t) = \langle \hbar k | \rho(t) | \hbar k' \rangle$ with $|\hbar k\rangle$ a momentum eigenstate. Expansion to first order in μ gives a decay of coherence in the momentum basis as the translation parameter increases.

4. Lorentz invariant formulation

In order to generalize these ideas to include Lorentz invariance, we must move to a field theory formulation. Space and time translations are determined by specifying the sources and detectors for the field. We are at liberty to choose any field at all, although in practice the electromagnetic field is the easiest to use. The source determines the fiducial state of the quantum field. Measurements reduce to particle detectors and space and time translation parameters are inferred by the statistics of detection events at such detectors. The relevant unitary translation generators are still position and momentum generators, but now constructed in the usual way from whatever quantum field we wish to use. As in the non-relativistic case, spatial translations require a fiducial state with an indefinite momentum while time translations require a fiducial state with and indefinite energy.

We will first discuss how space-time translations are determined in the standard formulation of quantum field theory. Estimation of a space-time translation in quantum parameter estimation theory was considered by Braunstein *et al* [1] and we now summarize that treatment. The generator for space-time translation is the energy-momentum 4-vector

$$\hat{\mathbf{P}} = \hat{P}^{\alpha}\mathbf{e}_{\alpha} = \hat{P}^{0}\mathbf{e}_{0} + \hat{\vec{P}} = \hat{P}^{0}\mathbf{e}_{0} + \hat{P}^{j}\mathbf{e}_{j}. \tag{38}$$

The space-time translation we seek to estimate can be written as

$$\mathbf{X} = S\mathbf{n} = Sn^{\alpha}\mathbf{e}_{\alpha}, \tag{39}$$

with

$$\mathbf{n} = n^{0}\mathbf{e}_{0} + \vec{n}, \tag{40}$$

is a space-like or time-like unit 4-vector specifying the direction of the space-time translation and S is the invariant interval that parameterizes the translation. The rotation of the fiducial state $|\psi_0\rangle$ in Hilbert space is then

$$|\psi_S\rangle = \mathrm{e}^{\mathrm{i}S\mathbf{n}\cdot\hat{\mathbf{P}}/\hbar}|\psi_0\rangle, \tag{41}$$

with

$$\mathbf{n}\cdot\hat{\mathbf{P}} = \eta_{\alpha\beta}n^{\alpha}\hat{P}^{\beta} = n^{\alpha}\hat{P}_{\alpha} = -n^{0}\hat{P}_{0} + \vec{n}\cdot\hat{\vec{P}}. \tag{42}$$

The Minkowski metric is $\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$ (we use units such that $c = 1$). The three-dimensional dot product is written as $\vec{n}\cdot\hat{\vec{P}} = n^{j}\hat{P}^{j}$. Braunstein *et al* [1] show that the parameter-based uncertainty principle for estimating the space-time translation parameter, S is

$$\langle(\delta S)^2\rangle_S\langle(\mathbf{n}\cdot\Delta\hat{\mathbf{P}}\rangle = \langle(\delta S)^2 n^{\alpha}n^{\beta}\langle\Delta\hat{P}_{\alpha}\Delta\hat{P}_{\beta}\rangle \geqslant \frac{\hbar^2}{4N}, \tag{43}$$

for N trials. When **n** is time like this is a time-energy uncertainty relation for the observer whose 4-velocity is **n**, and when **n** is space-like, this is a position-momentum uncertainty relation for an observer whose 4-velocity is orthogonal to **n**.

What fiducial states are appropriate for estimating a space-time translation? We will discuss the case of space and time translations separately to parallel the discussion in the non-relativistic case. For specificity, we will assume that we are using the electromagnetic field. It should be noted that this is a special case, but will suffice to illustrate the principles of the more general situation. The energy momentum 4-vector can be written most easily if we decompose the field into plane wave modes,

$$\hat{\mathbf{P}} = \sum_{\vec{k},\sigma} \hbar \mathbf{k} \hat{a}^{\dagger}_{\vec{k},\sigma} \hat{a}_{\vec{k},\sigma}, \tag{44}$$

where $\mathbf{k} = \omega \mathbf{e}_0 + \vec{k} = \omega \mathbf{e}_0 + k^j \mathbf{e}_j$ is a null wave 4-vector with $\omega = |\vec{k}| = k$ and the sum is over all wave 3-vectors $\vec{k}$ and polarization σ. The generator for space-time translations is thus determined by the number operator for the field modes which is the generator for phase shifts in the field. Thus, determining a space-time translation via the electromagnetic field reduces to phase parameter estimation. Optimal phase estimation is not a straightforward measurement, particularly in the multimode case [24]. However, it will suffice for our purposes to give a simple example based on photon counting. (Ultimately all field measurements reduce to counting field quanta.)

Let us consider just two modes, with wave 4-vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, with the same polarization. We will designate a Fock state for the mode $\mathbf{k}_i$ as $|n\rangle_i$. Suppose we have a source that produces the single photon state $|1\rangle_1 \otimes |0\rangle_2$, i.e. one photon in mode $\mathbf{k}_1$ and the vacuum in mode $\mathbf{k}_2$. The first step is to find a unitary transformation, U to give

$$U|1\rangle_1 \otimes |0\rangle_2 = \tfrac{1}{2}(|1\rangle_1 \otimes |0\rangle_2 + |0\rangle_1 \otimes |1\rangle_2). \tag{45}$$

If the modes have the same frequency, $\omega_1 = \omega_2$, this can be performed with a simple linear optical device known as a beam splitter, but if the modes also have different frequencies we need the nonlinear optical device known as a frequency converter. The state is now subjected to the unitary space-time translation in equation (41), followed by $U^{\dagger}$. The final state is

$$|\psi_S\rangle = \mathrm{e}^{\mathrm{i}S\delta_+/2} \left(\cos(S\delta_-/2)|0\rangle_1 \otimes |1\rangle_2 + \mathrm{i}\sin(S\delta_-/2)|1\rangle_1 \otimes |0\rangle_2\right), \tag{46}$$

where

$$\delta_{\pm} = \mathbf{n} \cdot (\mathbf{k}_1 - \mathbf{k}_2). \tag{47}$$

A simple measurement can now be made of the photon number difference between the two modes, with results ± 1 occurring with probabilities

$$P(+1) = 1 - P(-1) = \sin^2(S\delta_-/2). \tag{48}$$

Sampling this distribution enables an inference of the space-time translation parameter S. Of course such a measurement is not optimal. Using many photon states and a different kind of output measurement, it is possible to do much better [25].

If we seek only a space translation (i.e. a ruler) then we can choose the modes to have the same frequency, but wave vectors in different directions. Such a state clearly has an indefinite 3-momentum as we found for the non-relativistic case. If we seek a time translation (i.e. a clock) we must choose the wave vectors to have a different frequency, that is to say a different energy as in the non-relativistic case.

It is now straightforward to define an intrinsic decoherence model that is Lorentz invariant. The change in the state of a quantum field as a function of the displacement interval is

$$\frac{\mathrm{d}\rho(S)}{\mathrm{d}S} = \frac{1}{\nu}\left(\mathrm{e}^{\mathrm{i}\nu\mathbf{n}\cdot\hat{\mathbf{P}}/\hbar}\rho(S)\mathrm{e}^{-\mathrm{i}\nu\mathbf{n}\cdot\hat{\mathbf{P}}/\hbar} - \rho(S)\right). \tag{49}$$

Equivalently

$$\rho(S) = \sum_{m=0}^{\infty} \frac{(S/\nu)^m}{m!} \mathrm{e}^{-S/\nu} \mathrm{e}^{\mathrm{i}m\nu \mathbf{n}\cdot\hat{\mathbf{P}}/\hbar} \rho_o \mathrm{e}^{-\mathrm{i}m\nu\mathbf{n}\cdot\hat{\mathbf{P}}/\hbar}. \tag{50}$$

As the generator of space-time translations is already explicitly Lorentz invariant, these equations are Lorentz invariant. In fact $S\mathbf{n}\cdot\hat{\mathbf{P}}$ is nothing more than the action associated with the space-time interval. The central assumption for this relativistic model of intrinsic decoherence is that the action along some worldline can vary from trial to trial in an experimental determination of a space-time translation (this observation suggests an equivalent formulation in terms of path integrals). The state ρ is a many-particle field state and would typically be specified in the Fock basis for some mode decomposition. The specific form the intrinsic decoherence takes depends on the field under discussion through the energy momentum 4-vector. We now consider some consequences of this equation for the case of the electromagnetic field.

The most obvious modification is to the experimentally observed dispersion relation. The dispersion relation must be determined by making phase-dependent measurements on the field amplitude at different space-time points. We have postulated that such repeated measurements are described by a density operator $\rho(S)$ rather than a pure state, and we have given a rule for how to translate this state to describe measurements made at different space-time positions. In order to measure a field amplitude that is nonzero, we must specify a fiducial field state that has a nonzero amplitude. We will take this to be a coherent state [21].

In the standard theory of the electromagnetic field, we specify the electric field at position $\vec{x}$ by the operator

$$\hat{E}(\vec{x}) = \sum_k (u_k(\vec{x})a_k + u_k(\vec{x})^* a_k^\dagger), \tag{51}$$

where a_k, $a_k^\dagger$ are boson annihilation and creation operators, while $u_k(\vec{x})$ are a set of orthonormal mode functions and choose the state of the field on the space-like hypersurface $t = 0$ to be a coherent state such that

$$\mathrm{tr}[a_k\rho] = \alpha_k. \tag{52}$$

This is a semiclassical state for which the field amplitude on $t = 0$ is given by

$$\mathcal{E}(\vec{x}) = \sum_k (u_k(\vec{x})\alpha_k + u_k(\vec{x})^*\alpha_k^*). \tag{53}$$

We now translate the field along the time-like direction $n^\alpha = (1, 0, 0, 0)$, so that the field amplitude becomes

$$\mathcal{E}(\vec{x}, t) = \mathrm{tr}[\hat{E}(\vec{x})\rho(t)]. \tag{54}$$

Using equation (50) for this space-time path we have that

$$\rho(t) = \sum_{n=0}^{\infty} \frac{(t/\nu)}{n!} \mathrm{e}^{-t/\nu} \mathrm{e}^{-\mathrm{i}n\nu\sum_k k a_k^\dagger a_k} \rho(0) \mathrm{e}^{\mathrm{i}n\nu\sum_k k a_k^\dagger a_k}. \tag{55}$$

The field amplitude at $t \neq 0$ is then determined by

$$\mathrm{tr}\,[a_k\rho(t)] = \sum_{n=0}^{\infty} \frac{(t/\nu)}{n!} \mathrm{e}^{-t/\nu} \mathrm{tr}\left[a_k \mathrm{e}^{-\mathrm{i}n\nu\sum_j j a_j^\dagger a_j} \rho(0) \mathrm{e}^{\mathrm{i}n\nu\sum_j j a_j^\dagger a_j}\right] \tag{56}$$

$$= \sum_{n=0}^{\infty} \frac{(t/\nu)}{n!} \mathrm{e}^{-t/\nu} \mathrm{tr}\left[\rho(0) \mathrm{e}^{\mathrm{i}n\nu\sum_j j a_j^\dagger a_j} a_k \mathrm{e}^{-\mathrm{i}n\nu\sum_j j a_j^\dagger a_j}\right] \tag{57}$$

$$= \alpha_k \exp\left[\frac{t}{\nu}(\mathrm{e}^{-\mathrm{i}\nu k} - 1)\right]. \tag{58}$$

Thus

$$\alpha_k(t) = \alpha_k \mathrm{e}^{-\mathrm{i}\omega(k)t} \mathrm{e}^{-\gamma(k)t} \tag{59}$$

where the observed frequency of this mode amplitude is

$$\omega(k) = \frac{\sin(\nu k)}{\nu}, \tag{60}$$

and the amplitude decays due to intrinsic decoherence at the rate

$$\gamma(k) = \frac{1}{\nu}(1 - \cos(\nu k)). \tag{61}$$

This result is similar in form to what happens to a coherent state propagating in a Kerr medium (one with an intensity-dependent refractive index) [26]. As $\nu \to 0$, we recover the standard dispersion relation with a small modification

$$\omega(k) = k\left(1 - \frac{\nu^2 k^2}{6} + \cdots\right). \tag{62}$$

We are not restricted to preparation of coherent states to test this effect. A similar result is found when we consider single photon states, such as might be emitted in γ-ray bursts from cosmological distances [27]. A single photon state defined on the hypersurface at $t = 0$ is [28]

$$|1\rangle_\beta = \sum_k \beta_k a_k^\dagger |0\rangle. \tag{63}$$

The probability to detect a single photon per unit time, at space-time point $(\vec{x}, t)$ is then proportional to

$$n(\vec{x}, t) = \mathrm{tr}(\hat{E}^{(+)}(\vec{x})\hat{E}^{(-)}(\vec{x})\rho(t)), \tag{64}$$

where the positive frequency components of the field are defined by

$$\hat{E}^{(-)}(\vec{x}) = \sum_k u_k(\vec{x}) a_k. \tag{65}$$

Using equation (55) we find that

$$n(\vec{x},t)=\sum_{k_1,k_2}u^*_{k_1}(\vec{x})u_{k_2}(\vec{x})\beta^*_{k_1}\beta_{k_2}\exp\left[-\frac{t}{\nu}\left(1-\mathrm{e}^{\mathrm{i}\nu(k_1-k_2)}\right)\right]. \tag{66}$$

This is very different from the conventional result that can be found by setting $\nu\to 0$,

$$n_{con}(\vec{x},t)=\left|\sum_k u_k(\vec{x})\beta_k\mathrm{e}^{-\mathrm{i}kt}\right|^2, \tag{67}$$

note that the phase-shift term in equation (66) is similar to various quantum gravity predictions which suggest that the effect is as if the photon was propagating through a medium with a nonlinear refractive index. It is somewhat surprising that such a simple model as that presented here leads to modified dispersion relations similar to a number of models for quantum gravity [27], [29]–[31] with a very different starting point. However, the presence of the decay term, reflecting intrinsic decoherence, is not present in all of these theories.

5. Discussion and conclusion

We have proposed a model of intrinsic decoherence that is explicitly Lorentz invariant in so far as the generator of the unital semigroup representation is Lorentz invariant. Energy and momentum remain conserved quantities. The resulting theory makes specific predictions on the ultimate accuracy of spatial and temporal translations. In the case of the electromagnetic field the theory also predicts that an experimental determination of the dispersion relation will reveal a departure from the classical result. Similar results have been shown to occur in quantum gravity theories that explicitly violate Lorentz invariance. Unlike some other approaches, our theory does not use modified commutation relations for the generators of space-time translations [15, 32]. The violation of Lorentz invariance is only apparent due to an intrinsic uncertainty in space-time translations over multiple trials. This is similar to the quantum gravity models of Gambini *et al* [13].

How well do current experiments exclude such a theory? In the case of temporal translations, we need to consider the state-of-the-art for atomic clocks. The key issue here is the *instability* of a clock defined as a measure of the variation in the size of the intervals between clock ticks [36]. In the case of an atomic clock the instability is determined by the ratio of the line-width of the transition $\delta\omega$ to the frequency of the transition ω,

$$\sigma=\frac{\delta\omega}{\omega}. \tag{68}$$

For the very best atomic Cs fountains, this stability is $\sigma<4\times10^{-16}$. In section 3, we saw that the intrinsic decoherence model, in the limit $\nu\Delta_E\ll 1$, is equivalent to an atomic transition with intrinsic dephasing line-width of $\delta\omega=\nu\omega^2/2$. For the very best Cs atomic clocks (with $\omega=9.192\,\mathrm{GHz}$), $\nu<10^{-25}$ s in order to have escaped detection. As progress continues towards optical frequency standards based on cold neutral atom traps, we can expect fractional frequency instabilities at or below 10^{-17} [33], which at optical frequencies of $\omega \approx 10^{15}$ would give a constraint

$$\nu < 10^{-32}\,\mathrm{s}. \tag{69}$$

However, the ultimate constraint might come from attempts to build a quantum computer using trapped ions. Already there have been suggestions on how to use quantum logic techniques from this field for precision spectroscopy [34]. There would be some irony in building a quantum computer to test quantum mechanics to destruction.

We now turn to the possible experimental tests based on deviations from the linear dispersion relation for the electromagnetic field. Various quantum gravity theories also predict a modified dispersion relation for the electromagnetic field and there is already some debate in the literature on possible experimental tests [35]. Here, we are primarily concerned with photons. Amelino-Camelia *et al* [27] has suggested that it should be possible to observe such effects from γ-ray bursts emitted from sources at cosmological distances.

We can assess this proposal using the single photon detection probability in equation (66). This result tells us the probability to detect a single photon at time t and point $\vec{x}$, given that we have prepared a 'global' single photon state on the space-like hypersurface at t_0. The detector of course is on the same space-like hypersurface. In γ-emission the detector is some distance from the source and it would be more appropriate to use a spherical wave mode decomposition to describe a point source. However, the additional spatial translation and more complicated mode functions do not change the fundamental result. We set $\vec{x} = 0$ for simplicity and given a plane wave mode function decomposition for the field, the detection probability is then given by

$$n(t) = \gamma \int_{-\infty}^{\infty} \mathrm{d}k_1\, \mathrm{d}k_2\, \beta_{k_1}^{*} \beta_{k_2} \exp\left[-\frac{t}{\nu}(1 - \mathrm{e}^{\mathrm{i}\nu(k_1 - k_2)})\right], \tag{70}$$

where γ is a geometric factor and we have taken a continuum approximation. This may be written as

$$n(t) = \gamma \sum_{n=0}^{\infty} P_n(t/\nu) |\beta(n\nu)|^2, \tag{71}$$

where $\beta(t)$ is the Fourier transform of the amplitude function β_k,

$$\beta(t) = \int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kt} \beta_k \tag{72}$$

and

$$P_n(t/\nu) = \frac{(t/\nu)^n}{n!} \mathrm{e}^{-t/\nu}, \tag{73}$$

is a Poisson probability distribution with mean t/ν. The conventional result, equation (67) is then seen to arise in the limit $\nu \to 0$, for which the Poisson distribution is very sharply peaked at $n = t/\nu$. The expression for $n(t)$ in equation (71) then has an obvious interpretation. The function $|\beta(t/\nu)|^2$ is the conventional transform limited expression to count a photon after a time $n\nu$. We can then regard $P_n(t/\nu)$ as the conditional probability to count a photon at time t, given a n (unobserved) Hilbert space phase shifts. The net effect is to broaden the observed temporal width of the photon over what it would be for a transform-limited pulse. This means that, in many trials with identically prepared photo-emission events (that is to say, the same β_k), some photons will arrive *sooner* than expected in conventional quantum optics (and some will arrive later). This assumes of course that the temporal width of $|\beta(t)|^2$ is smaller than the width of the Poisson distribution, $P_n(t/\nu)$, which is given by $\sigma = \sqrt{t/\nu}$. If some reasonable assumptions can be made about the form of β_k from the likely emission processes, this might enable an experimental test of the theory. The key signature is to show that the effect becomes more significant the greater is the transit time of the photon. Unfortunately, there are many reasons why a single photon pulse will suffer phase diffusion and deviate from transform-limited behaviour. This simply reflects the fact that to verify any mechanism for intrinsic decoherence, we must first eliminate or control for all the many sources of extrinsic decoherence that abound in this world. Maybe for sufficiently high-energy photons, this will be possible.

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