

# Clock Synchronization without Frequency References under Weak Timing Correlations

Justin Yu Xiang Peh,<sup>1</sup> Darren Ming Zhi Koh,<sup>1</sup> Zifang Xu,<sup>2</sup> Xi Jie Yeo,<sup>1</sup> Peng Kian Tan,<sup>1</sup> and Christian Kurtsiefer<sup>1,2</sup>

<sup>1</sup>*Centre for Quantum Technologies, 3 Science Drive 2, Singapore 117543*

<sup>2</sup>*Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore, 117542*

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**Clock synchronization is necessary for communication and distributed computing tasks. Previous schemes based on photon timing correlations use pulsed light or photon pairs for their strong timing correlations. In this work, we demonstrate successful synchronization of quartz clocks using weakly time-correlated photons of 180 ns coherence time from a pseudothermal bunched light source. A synchronization timing jitter of 10 ns is achieved over symmetric -102 dB optical loss channels between two parties, over a span of 25 hours. We also derive a model to accurately estimate the coherence peak searching success probabilities.**

## I. INTRODUCTION

Clock synchronization is used in everyday tasks such as navigation and distributed computing. This is commonly implemented using the Network Time Protocol or global navigation satellite system (e.g. GPS) time synchronization, achieving precision of milliseconds or tens of nanoseconds respectively [1, 2]. Quantum communication protocols also require clock synchronization, but on the order of nano- to pico-seconds, and is typically achieved either using pulsed laser light [3] or with classical signals [4, 5].

Modern systems can use the resources typically in the protocol itself to perform the clock synchronization, such as photon pairs from spontaneous parametric down-conversion (SPDC) [6]. This is possible with the use of frequency standards such as Rubidium (Rb) clocks which provide a long term frequency stability of <1 ppb/day. In comparison, crystal oscillators without temperature stabilization have frequency stability of only 100 ppb/day.

To remove the dependency on Rb frequency standards, it was proposed to find and track the timing difference between photon pairs due to their strong timing correlation on the order of picoseconds [7, 8], by identifying an initial coincidence peak with low timing resolution then iteratively apply frequency corrections to improve resolution. Weak coherent photon pulses were also proposed for clock frequency transfer [9]. In both cases, the cross-correlation peak is very strong, i.e.  $g^{(2)}(\tau = 0) \gg 2$ , with resolution generally limited only by the timing jitter of the generation and detection optics.

Another source of timing correlated light are thermal

photons which has been used in application such as ghost imaging [10] and range finding [11]. These timing correlations arise from temporal photon bunching, also known as the Hanbury-Brown-Twiss effect [12]. In particular, the use of thermal light opens up the potential for distributed clock synchronization due to photon bunching being preserved across arbitrary partitioning.

However, unlike photon pairs, identification of this peak is more challenging due to the low signal (i.e.  $g^{(2)}(\tau) \leq 2$ ) exacerbated by background fluctuations. There is an earlier proposal that suggests the use of thermal light for clock synchronization, relying on the use of low efficiency two-photon absorption in single photon detectors to resolve the bunching characteristic [13, 14]; such a scheme has yet to be demonstrated.

Here, we demonstrate clock synchronization using one such weak timing correlation light source, i.e. light from pseudothermal source with a  $g^{(2)}(\tau) = 1.44$  and coherence time  $\tau_c = 180(6)$  ns, over a -102 dB transmission channel with an accuracy of 10 ns, and continuously track the frequency drift over a period of 25 hours to demonstrate its stability. Our scheme only requires single photon detection and crystal oscillators as reference clocks. We additionally derive the probability of achieving clock synchronization over transmission channels, which is applicable across different time-correlated photon sources, including SPDC pair sources and thermal sources.

## II. CLOCK SYNCHRONIZATION WITH BUNCHED LIGHT

Timing synchronization was achieved using bunched light between two remote parties over symmetrical loss channels, each with -102 dB average transmission, for more than 25 hours using only simple crystal oscillators, see Fig. 1.

The source of bunched light is a pseudothermal source [15], established using a 780 nm laser sent into an unbalanced Mach-Zehnder interferometer with optical delay longer than the laser coherence time  $\tau_c$ ; phase fluctuations in the laser manifests the bunching effect, see Appendix A for a detailed explanation. This results in an intensity correlation that can be measured through the second-order coherence function  $g^{(2)}(\tau)$ , given by

$$g^{(2)}(\tau) = 1 + \frac{1}{2} \exp\left(-\frac{2|\tau|}{\tau_c}\right) \quad (1)$$

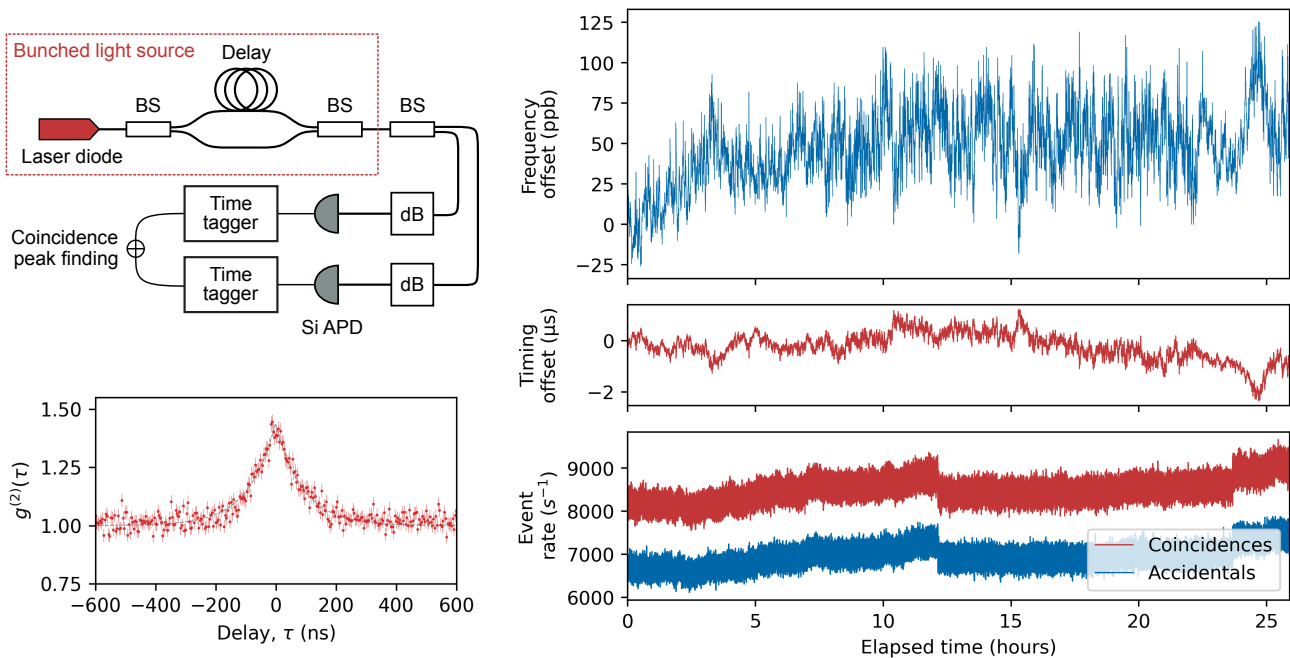


FIG. 1. **Top left:** Simplified experimental setup for clock synchronization. Bunched (pseudothermal) light is sent to two different parties through separate  $-102$  dB channels, before detection by Silicon avalanche photodetectors (Si APDs) and timestamping by time taggers running on independent quartz clocks. Singles count rate  $s_1$  and  $s_2$  of approximately 200 kcounts/s are recorded on each side. BS: beamsplitter; dB: attenuators.

**Bottom left:** Plot of  $g^{(2)}(\tau)$  of the pseudothermal light source, with the curve fit in grey corresponding to  $g^{(2)}(0) = 1.44(1)$  peak and coherence time  $\tau_c = 180(6)$  ns. Error bars correspond to Poissonian errors from counting statistics.

**Right:** Long-term trace of frequency offset, timing offset, and event rates during a 25 hour clock synchronization run between independent time taggers running on separate quartz clocks, after an initial frequency correction of 4.0 ppm. Correlation peak tracking is performed using a 256 ns coincidence window. The sharp spike and dip in event rates are attributed to laser mode hopping.

due to the Lorentzian spectral profile of the laser, where the coherence time  $\tau_c$  is inversely proportional to the frequency spectral width of the source  $\Delta f$  [16]. The brightness of the source is  $1.55(2)$  mW, with a measured coherence peak of  $g^{(2)}(\tau = 0) = 1.44(1)$  and coherence time of  $\tau_c = 180(6)$  ns.

The light from the source is shared between  $n = 2$  parties with optical fiber channel transmissions of  $-101.9(4)$  dB and  $-102.2(4)$  dB using optical attenuators. Photon arrival events are detected using Silicon avalanche photodetectors (APDs) on each side, with count rates of 192 kcounts/s and 182 kcounts/s without correcting for dark counts and afterpulsing.

The detection events are read by independent time taggers disciplined by different free-running 10 MHz quartz clocks. The generated timestamps are then exchanged for coincidence peak finding and tracking in real-time. The peak tracking program continuously serves a timing offset between each party by pairing photon detection events between both parties within a 256 ns coincidence window, over the span of the 25 hour measurement. The average coincidence rate of 8,500 events/s is consistently higher than the accidental count rate of 7,000 events/s, indicative of successful frequency tracking.

Due to the presence of clock frequency drift, typically from temperature fluctuations and electronic noise, the frequency offset between the clocks changes over time as well. We are able to reconstruct this offset, by monitoring the drift in served timings using individual samples spanning 10.74 s, with a resolution of 0.537 s, shown in Fig. 1.

The actual frequency offset was obtained via a heterodyne measurement using identical copies of the clock signals with integration time of 10 s, shown in Fig. 2. A maximum instantaneous frequency difference of 35.4 ppb, corresponding to an average drift of 3.3 ppb/s, was observed.

We estimate the error in the reconstructed signal by performing linear interpolation on the measurement and subsequent differencing. The reconstructed frequency offset is found to be in good agreement with the measured frequency offset, with a root-mean-squared error of 3.2 ppb averaged over the 25 hour span.

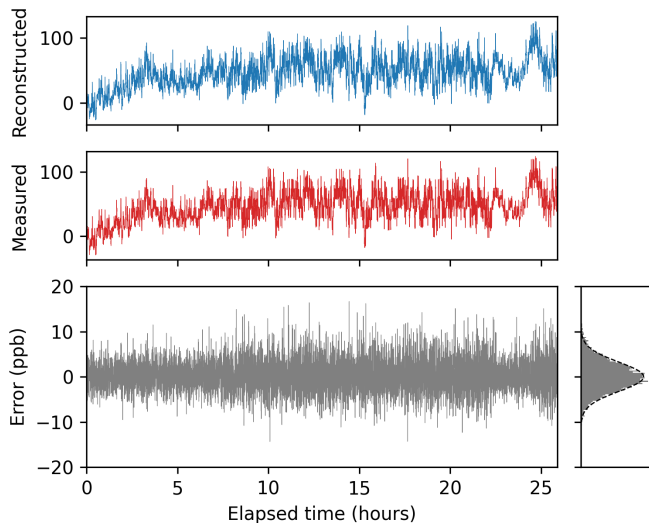


FIG. 2. A concurrent measurement of the actual frequency offset between the two external 10 MHz clocks, as well as the corresponding error in the reconstructed frequency offset. The accuracy of the frequency servoed by the peak tracking algorithm is 3.2 ppb on average.

### III. PEAK SEARCHING

Clock synchronization can be decomposed into two parts: first identifying the initial clock frequency and time difference between two parties, then tracking of said peak to monitor the timing drift. The former is performed by distributing photons with time-correlated statistics to each party, and finding the coincidence peak using cross-correlation to identify the time delay.

Efficient peak identification relies on the circular convolution theorem to compute the cross-correlation between two sets of detection timestamps  $a[k]$  and  $b[k]$ ,

$$g^{(2)}(\tau) \sim (a \star b)[k] = \mathcal{F}^{-1} \left\{ \overline{\mathcal{F}\{a\}} \cdot \mathcal{F}\{b\} \right\} [k], \quad (2)$$

under the discretization  $\tau = k\delta t$  with  $k \in \mathbb{Z}$  and time resolution  $\delta t$ , using the Fourier Transform  $\mathcal{F}$  and its inverse (see Appendix B). The success of peak identification is constrained by the singles rate and the rate of true coincidences, and additionally requires the choice of optimal parameters for the Fast Fourier Transform (FFT) parameters for computing the time delay: specifically, the number of time bins  $N = 2^q$  ( $q \in \mathbb{Z}^+$ ), and initial bin width  $\delta t$ .

We model the peak finding probability — in both cases, with and without a reliable frequency reference between both parties — and perform an exhaustive parameter scan to identify appropriate FFT parameters in Fig. 3. Our model provides better estimations of the peak finding probability compared to previous works, by avoiding the normal approximation to the noise in the FFT (see Appendix C).

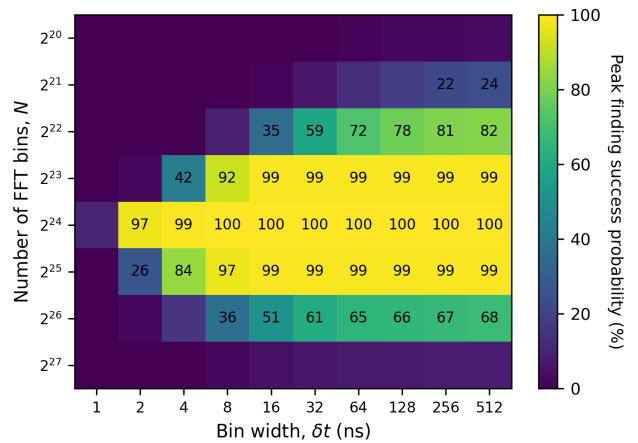


FIG. 3. Simulated probabilities to find the correct peak position by solving the required true coincidence equation found in Eqn. 5, given the singles detection rate  $s_1 = s_2 = 100$  kcounts/s, coincidence rate  $c = 650$  counts/s, bin overlap  $\nu = 0.5$ , and frequency offset error of up to  $\Delta u = 100$  ppb. These parameters correspond to the setup in Fig. 1 with an additional 3 dB attenuation per channel, and shows good agreement with measurements.

#### A. No frequency offset

The minimum acquisition time required for the cross-correlation is  $T = N\delta t$  to obtain a flat cross-correlation noise floor. This noise floor arises from accidental coincidences — attributed to noise sources, such as dark counts, and coincidences between uncorrelated detection events — with an expected value of  $C_a = (s_1 s_2 \delta t) T$ , where  $s_1$  and  $s_2$  are the detection event rates on each channel. The detection events are well-approximated by a Poisson distribution after binning [17], so the accidental coincidences in each time bin also follows a Poisson distribution, i.e.  $X_{k \in \{1, \dots, N\}} \sim \text{Poisson}(\lambda = C_a)$ .

The maximum observed value across all time bins  $X_{(N)} \equiv \max\{X_k\}$  is therefore given by the max-order Poisson distribution (derived in Appendix D) whose probability distribution is

$$f_{X_{(N)}}(x) = [F_X(k|\lambda)]^N - [F_X(k|\lambda) - f_X(k|\lambda)]^N$$

where  $f_X$  and  $F_X$  respectively correspond to the probability mass and cumulative distribution functions of a single bin.

The coincidence rate above background accidentals in a single time bin (denoted  $c_e$ ) required to be identified as the highest peak in the cross-correlation is thus

$$c_e > \frac{1}{T} (X_{(N)} - X). \quad (3)$$

The coincidence rate per time bin can be maximized by setting the timing resolution  $\delta t$  to be of the same scale as the coherence time  $\tau_c$ , so that most of the coincidence events fall within the same time bin, i.e.  $\delta t \sim \tau_c$ . Some

of these events may fall into an adjacent time bin instead due to off-centered bins (since the exact time offset is not known *a priori*): this introduces a smudge factor into Eqn. 3 representing the degree of bin overlap  $\nu \in [0.5, 1]$ , yielding

$$c_e > \frac{1}{\nu T} (X_{(N)} - X). \quad (4)$$

### B. Variable frequency offset

In the case of two separate clocks with slightly different clock frequencies, the timing delay between photon arrivals between each party can drift by  $\Delta\tau_{\text{drift}} \approx T\Delta u(t)$  due to the non-zero clock-frequency offset  $\Delta u(t)$  after some elapsed time  $T$ . Under a sufficiently small  $T$ , the frequency offset can be approximated as a constant, i.e.  $\Delta u(t) = \Delta u$ .

In order to maximize the number of coincidences in a single time bin, the time bin should ideally be as wide as the timing drift, i.e.  $\delta t > \Delta\tau_{\text{drift}}$ , or in other words,  $N < 1/\Delta u$ . We model this as an additional smudge factor  $\mu \equiv \max\{1, N\Delta u\}$ , and together with Eqn. 4 yields the minimum required true coincidence rate for successful peak finding given by

$$c_e > \frac{1}{\mu\nu T} (X_{(N)} - X). \quad (5)$$

The surface of this equation is plotted in Fig. 3, by performing Monte Carlo simulations for a specific set of parameters and across different  $N$  and  $\delta t$ .

The reduction in coincidence counts due to the presence of frequency offset can be mitigated either by choosing a smaller  $N$ , or by performing a frequency precompensation [8] on the set of timestamps  $\{t_i\}$  to reduce the apparent  $\Delta u$  between the two clocks and allow for larger  $N$  values. The compensation is given by the mapping

$$t_i \rightarrow \Delta t_i(1 + \Delta u) + t_{i-1} = t_i + \Delta t_i\Delta u, \quad (6)$$

where  $\Delta t_i \equiv t_i - t_{i-1}$  is the separation between consecutive timestamps.

In practice, while the frequency offset between quartz clocks can be high ( $\sim 10$  ppm), the short-term stability of the clocks themselves are much higher ( $< 10$  ppb/s), so after an initial precompensation reference, a small frequency precompensation step size of 100 ppb is sufficient to identify most peaks within 3 peak searching attempts [18].

Once a initial peak has been found, the frequency and timing resolution can be further improved by repeating the respective corrections with progressively smaller time bins, until the desired resolution has been reached [7]. Our optimized implementations of peak finding in Python, and frequency compensation in C, are available under GPL license in GitHub [19].

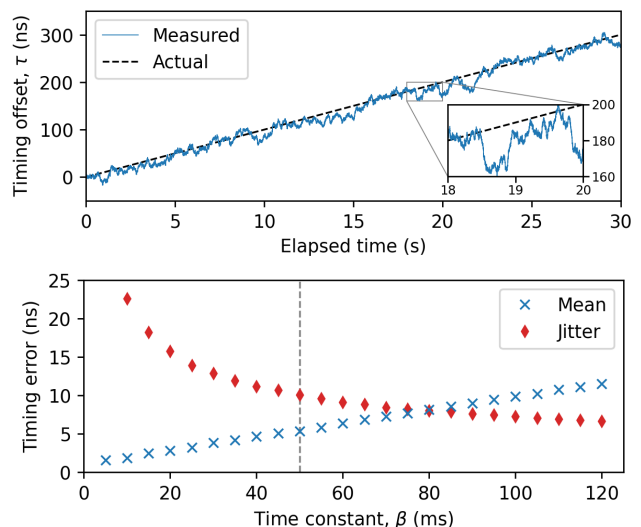


FIG. 4. Characterization of peak tracking accuracy under a constant frequency offset  $\Delta u = 10$  ppb, with a time constant  $\beta = 50$  ms chosen for the exponential moving average filter used in the experiment.

**Top:** Measurement of reported timing offsets (blue) against the actual offset (black) over a period of 30 seconds. The inset highlights a  $\sim 30$  ns timing error over a 0.5-second window, which suggests that a histogram fit does not improve accuracy.

**Bottom:** Measurements of average offset error and jitter across different time constants  $\beta$ , over a measurement period of 10 minutes. Timing jitter is equivalent to the root-mean-squared error, and corresponds to 10 ns with  $\beta = 50$  ms (marked in grey dashed line). Peak tracking fails with time constants less than 5 ms.

## IV. PEAK TRACKING

The use of quartz crystal oscillators as frequency references result in changes in frequency offset over time, due to their long-term  $\Delta u(t)$  stability of up to 100 ppb/day. This causes a drift in the correlation peak position over time which needs to be tracked.

Tracking can be performed by looking for coincidences within a sufficiently wide coincidence window, such that drifts in the peak are captured. However, when using bunched light for peak tracking, coincidences events are effectively dominated by background accidentals due to the low  $g^{(2)}(\tau)$  of the light source. Directly returning each time difference  $\tau_i$  found will result in the peak being quickly lost.

We apply a smoothing operation using an exponential moving average filter, to introduce damping against events far from the current estimated peak position. This effectively minimizes noise fluctuations and allows for the peak to be tracked.

In order to quantify the accuracy of the peak tracking, we supply a constant 10 ppb frequency offset between the two timestamps using a function generator and derive the true timing offset. Using a time constant  $\beta = 50$  ms for

the moving average filter, a timing jitter of 10 ns can be obtained, as shown in Fig. 4, which is an order of magnitude better than the  $\tau_c = 180(6)$  ns coherence time of the source. The jitter was tested to remain the same even under larger frequency offsets of 50 ppb as well.

A tracking lag is also present and can be seen in Fig. 4, where the measurement best-fit line is slightly displaced from the true timing offset. This is attributed to the frequency offset being non-zero and constant, and manifests as a mean timing error of around 6 ns at  $\beta = 50$  ms. While this lag scales proportionally to the frequency offset, this lag is expected to not contribute significantly due to the random fluctuations of the frequency offset about its mean, in the case of free-running clocks.

We also make a small note regarding the use of timing offset histograms to improve timing resolution. For photon pair and pulsed sources, higher timing resolution can be achieved by collecting a histogram within a small timing window and extracting the peak from a normal distribution curve fit, due to the narrow peak coherence signature of  $\tau_c \ll 1$  ns. However, for the bunched source with a significantly noisier coherence peak signature, this technique does not improve the timing jitter, as can be seen in the served timing offset in the inset of Fig. 4 being consistently far from the mean.

Active frequency compensation is also performed online to keep the frequency offset small, to avoid peak tracking loss since the correlation peak is less likely to drift out of the coincidence window. We do this by estimating the frequency offset  $\Delta u$  from the rate of change in timing drift (see Appendix E), then performing timing compensation for each timestamp using the estimated frequency offset, identical to that of Eqn. 6.

## V. CONCLUSION

We demonstrated successful clock synchronization between two parties over a symmetrical -102 dB transmission channel, using a pseudothermal bunched light source of coherence time  $\tau_c = 180(6)$  ns. Peak tracking was performed online over a span of 25 hours with active frequency compensation and an exponential moving average filter of time constant  $\beta = 50$  ns, achieving an overall timing jitter of 10 ns.

Previous papers on clock synchronization performed using photon pair sources rely on a peak significance metric for peak finding [7, 8], used as a threshold for

quantifying the probability of the cross-correlation peak being attributed to noise. While it remains a useful metric for estimating peak location, this underestimates the peak finding probabilities under low signal conditions with small time bins. We develop a model that accounts for the Poisson nature of the coincidences, and calculate instead the true probability of the cross-correlation peak being the signal, as well as derive optimal FFT parameters for a given true coincidence and singles rate. This remains applicable even when using any other sources of timing correlations, including photon pairs and thermal light.

The clock synchronization scheme in this work can be directly applied to protocols that use the photon bunching as a resource, or indirectly by compensating the reported timings with the measured timing and frequency offsets. Direct frequency compensation can also be performed by actively correcting the quartz oscillator frequencies, e.g. by means of a voltage error signal, so that the clock signals themselves can be utilized as part of a clock distribution network.

We can also take advantage of the fact that the second-order coherence is preserved across arbitrary partitioning of the light, to distribute the signal amongst multiple parties in a star topology. Since splitting light into two separate channels incurs an additional insertion loss of 3 dB per channel, clock distribution to  $2^n$  parties can be achieved with only  $-3n$  dB of additional loss per channel, e.g. a 128-party setup with this source incurs about  $-80$  dB.

This work paves the way towards clock synchronization using telecommunication O-band and C-band bunched light, which will be able to propagate with minimal chromatic dispersion over longer distances, e.g. 500 km of G.652 telecommunication fiber with 0.2 dB/km optical loss at 1550 nm. This additionally opens up the possibility of using erbium-doped fiber amplifiers to amplify the correlation signal. While timing offset changes as a result of thermal expansion in optical fibers will need to be addressed, this effect is fundamentally indistinguishable from a clock frequency drift in this scheme and can hence be compensated for in the same manner.

## VI. ACKNOWLEDGEMENTS

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## Appendix A: Pseudothermal source and experimental setup

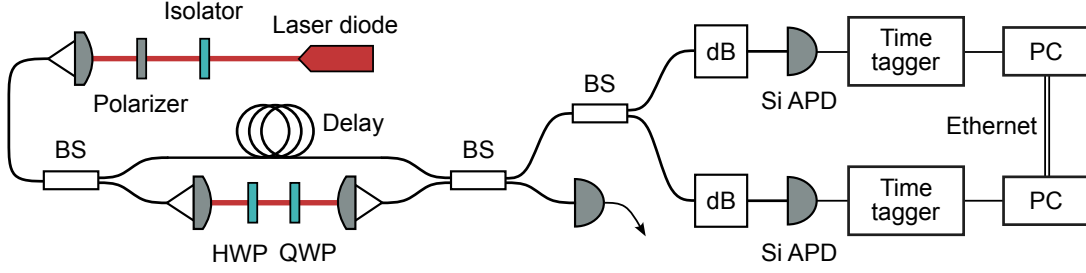


FIG. 5. Detailed experimental setup using a pseudothermal bunched light source. BS: 50:50 beamsplitter; dB: attenuator; HWP: half-wave plate; QWP: quarter-wave plate; APD: avalanche photodiode.

### 1. Optical setup

The primary source used in the frequency tracking measurement is a pseudo-thermal light source, where light from a laser running above the lasing threshold is split into two separate paths, one of which is delayed beyond the coherence time of the laser. If the two paths are perfectly indistinguishable (i.e. in polarization, spatial mode, and brightness), they act as independent light emitters and exhibit bunching effects.

From the Wiener-Khinchine theorem [20], the Lorentzian spectral profile from a coherent laser light source corresponds to a  $g^{(1)}$  with a Laplacian timing profile of the form,

$$g^{(1)}(\tau) := \frac{\langle E^*(t)E(t+\tau) \rangle}{\langle E^*(t)E(t) \rangle} = \exp\left(-\frac{|\tau|}{\tau_c}\right)$$

as a function of coherence time  $\tau_c$ .

Thermal light can be modelled as a large collection of independent emitters of light [16]. In the case of  $\nu$  emitters, we have

$$\langle E^*(t)E(t+\tau) \rangle = \nu \langle E_i^*(t)E_i(t+\tau) \rangle$$

$$\begin{aligned} & \langle E^*(t)E^*(t+\tau)E(t+\tau)E(t) \rangle \\ &= \nu \langle E_i^*(t)E_i^*(t+\tau)E_i(t+\tau)E_i(t) \rangle \\ &+ \nu(\nu-1) [\langle E_i^*(t)E_i(t) \rangle + |\langle E_i^*(t)E_i(t+\tau) \rangle|] \end{aligned}$$

and we can recover the Siegert relation [21] and determine the  $g^{(2)}$  of the light source

$$\begin{aligned} g^{(2)}(\tau) &:= \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t) \rangle}{\langle E^*(t)E(t) \rangle^2} \\ &= 1 + \frac{\nu-1}{\nu} |g^{(1)}(\tau)|^2 \\ &= 1 + \frac{1}{2} \exp\left(-\frac{2|\tau|}{\tau_c}\right), \quad \nu = 2 \end{aligned}$$

which saturates at a value of  $g^{(2)}(\tau=0) = 1.5$ .

Our experimental setup, per Fig. 5, uses light from a distributed-feedback (DFB) laser diode of wavelength 780 nm, coupled into 780-HP fiber to project into a single optical mode. The coherence length of the laser is approximately

200 m; a delay fiber of length 400 m was used in the delay arm, corresponding to a 2  $\mu$ s delay. Polarization rotation in the delay arm is compensated using a free-space link with a HWP and QWP, while attenuation is achieved by slight decoupling of the collection mode, before recombination using a fused-coupler beamsplitter (BS). The  $g^{(2)}(\tau = 0) = 1.44 \pm 0.03$  achieved in the setup is attributed to the fact that the laser is not fully coherent [22] and other experimental imperfections.

A second BS is used to further split the light into two symmetrical channels for downstream detection using fiber-pigtailed active-quenched Si avalanche photodiodes (Excelitas SPCM-800-10-FC). Attenuation in each channel is achieved by cascading fiber beamsplitters as well as fiber decoupling at the mating sleeves, and is measured to be relatively stable over multiple days. Fiber spools for delay was not used to avoid potential complications from timing delays attributed to fiber length changes.

Time tagging is subsequently performed by timestamp devices (S-Fifteen Instruments TDC2) with 4 ps nominal timing resolution and 20 ps  $1\text{-}\sigma$  timing jitter. The clock to each timestamp device is supplied by external 10 MHz clock distribution boards without any onboard temperature stabilization.

## 2. Software setup

The timestamps are fed via a USB 2.0 uplink to two separate computers, which supplements coarse timing accuracy using the Network Time Protocol (NTP) on the order of milliseconds. The two computers rely on different master NTP servers for timing synchronization to emulate more realistic network conditions. Peak finding and tracking is subsequently performed to obtain a time difference with nanosecond resolution. The bidirectional communication for coincidence peak agreement is performed over direct TCP/IP sockets, using `qcrypto` with the `fpfind` [19] add-on for inline software-based active frequency compensation. Each pairwise event corresponds to a timing difference with nanosecond resolution.

The software responsible for peak tracking in `qcrypto` is `costream`, which was supplied with the parameters `-w 1024 -u 1024 -Q 50000 -a 4096` that accounted for the relatively broad coincidence peak. `costream` performs a two-pointer search for pairwise coincidence events, within a coincidence tracking window of width 256 ns to account for frequency drifts.

## 3. Concurrent frequency measurement

The actual frequency offset between the two clocks were measured concurrently with the clock synchronization experiment by mixing separate copies of the 10 MHz signals and passing through a low pass filter with 4 GHz cut-off frequency. This mixed signal was sampled by an oscilloscope at a rate of 2.5 ksamples/s over 10 s, before performing a  $2^{14}$ -bin FFT (nominal resolution of  $\sim 0.15$  Hz) with a Hann window.

The frequency offset error is then calculated by measuring the difference between the servoed frequency offset (from the frequency estimation step in the clock synchronization) and the measured frequency offset with linear interpolation. The histogram of offset errors is fitted using a Gaussian probability density function, obtaining a standard deviation of 3.2 ppb.



### Appendix B: Using FFT for cross-correlation

Identification of the timing delay between two parties involves histogramming of the time difference between all timestamps measured by both parties, and then identifying the cluster of time difference. After binning in the histogram, this appears as a peak in the histogram. For a set of  $m$  and  $n$  timestamps,

$$(f \star g)[n] = \sum_m f[m]g[n + m] \quad (\text{B1})$$

This has a time complexity of  $\mathcal{O}(mn)$ .

A more computationally efficient method to calculate the time lag relies on the circular convolution theorem to compute the discrete cross-correlation using the Fourier Transform, i.e.

$$(f \star g)[n] = \mathcal{F}^{-1} \left\{ \overline{\mathcal{F}\{f\}} \cdot \mathcal{F}\{g\} \right\} [n] \quad (\text{B2})$$

In discrete space, the discrete Fourier Transform can be efficiently implemented as a Fast Fourier Transform (FFT), which determines the final time complexity of the cross-correlation of  $\mathcal{O}(n \log n)$ . This is more efficient for larger number of elements  $n$ , at the expense of  $\mathcal{O}(n)$  memory.

We note that the histogramming method works well when the number of elements to be computed is small, by containing the search to a limited coincidence window such that  $m$  is as small as possible.

We additionally note that algorithms faster than  $\mathcal{O}(n \log n)$  time complexity are also possible [23, 24], but require additional encoding of time-synchronization strings within the light pulses.

### Appendix C: Peak finding probabilities using Poisson statistics vs Gaussian assumption

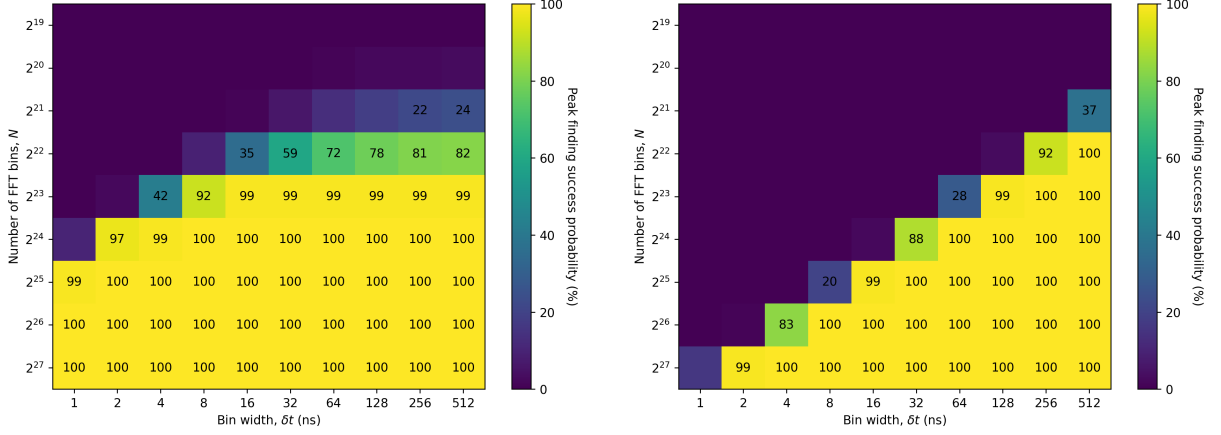


FIG. 6. **Left:** Probabilities to find the correct peak position by solving Eqn. 4, given the singles detection rate  $s_1 = s_2 = 100$  kcounts/s, coincidence rate  $c = 650$  counts/s, and bin overlap of  $\nu = 0.5$ . Since there is no frequency offset, the peak finding probabilities reaches 100% with increasing number of FFT bins.

**Right:** Probabilities to find the correct peak position using the normal approximation for  $X_i$  in Eqn. 4, with the same parameters. These

The corresponding probabilities for Poisson model and the corresponding Gaussian approximation model are presented in Fig. 6. We find that the previous models assuming Gaussian-distributed bin values incorrectly predict the success probability of peak finding: in particular, it generally underestimates the probability for low time bin widths, and overestimates for small number of FFT bins.

FFT can be efficiently implemented if the number of bins  $N$  are either in powers of 2, or has factorizations with small primes. Here we use  $N = 2^q, q \in \mathbb{Z}^+$  for its logarithmic scale.

### Appendix D: Derivation of max-order distribution

Given that the FFT is performed with  $N$  bins, the accidental coincidences in each of these bins follow the Poisson distribution with mean  $\lambda$ . The maximum value across all these bins increases with an increasing number of bins. The distribution of the max-order statistic (i.e. the maximum of all bins) with respect to the number of coincidences  $k$  and number of bins  $N$  of mean value  $\lambda$  is denoted  $f_{(N)}(k|\lambda)$ .

Since each bin is assumed to be independent and identically distributed, the corresponding max-order cumulative distribution function (CDF) can be expressed as the product of the CDF of individual bins,

$$F_{(N)}(k|\lambda) = \underbrace{F(k|\lambda) \times \dots \times F(k|\lambda)}_{N \text{ times}} = [F(k|\lambda)]^N.$$

We therefore express the max-order probability mass function (PMF) in terms of the Poisson PMF and CDF of a single bin,

$$\begin{aligned} f_{(N)}(k|\lambda) &= F_{(N)}(k|\lambda) - F_{(N)}(k-1|\lambda) \\ &= [F(k|\lambda)]^N - [F(k-1|\lambda)]^N \\ &= [F(k|\lambda)]^N - [F(k|\lambda) - f(k|\lambda)]^N. \end{aligned}$$

Notably, this form is tenable for direct computation without additional simplification, even though computing the difference of  $n^{\text{th}}$ -powers generally causes catastrophic cancellation due to floating-point rounding errors. The max-order Poisson PMF distribution width heuristically scales roughly with  $\sqrt{\lambda}$ , which requires the PMF to be accurate to at least  $1/\sqrt{\lambda}$ , e.g.  $\lambda \leq 10^4$  needs at least  $\sim 10^{-2}$  accuracy. Since  $F(k|\lambda) \in [0, 1]$  and the fact that exponentiation can be easily applied for large  $N$  using numerical techniques (such as exponentiation-by-squaring), the rounding errors can be minimized to near the floating-point precision, e.g.  $\sim 10^{-16}$  for 64-bit floats.

For larger  $\lambda > 10^4$  (i.e. high accidental coincidence rates per time bin), the normal approximation for each bin remains appropriate,

$$f(k) \sim \text{Poisson}(k|\lambda) \approx \mathcal{N}(x = k | \mu = \lambda, \sigma = \sqrt{\lambda}),$$

and the corresponding max-order probability distribution function (PDF) follows a more tractable form for numerical computation,

$$\begin{aligned} f_{(N)}(x|\mu, \sigma) &= \frac{d}{dx} F_{(N)}(x|\mu, \sigma) \\ &= \frac{d}{dx} [F(x|\mu, \sigma)]^N \\ &= N f(x|\mu, \sigma) [F(x|\mu, \sigma)]^{N-1}. \end{aligned}$$

This in fact corresponds to the first order term for the discrete case after a binomial expansion, noting that  $f(k|\lambda) \leq F(k|\lambda)$ , with

$$\begin{aligned} f_{(N)}(k|\lambda) &= [F(k|\lambda)]^N - [F(k|\lambda) - f(k|\lambda)]^N \\ &= [F(k|\lambda)]^N - \left[ [F(k|\lambda)]^N - N f(k|\lambda) [F(k|\lambda)]^{N-1} + \dots \right] \\ &= N f(k|\lambda) [F(k|\lambda)]^{N-1} - \mathcal{O}(f^2 F^{N-2}). \end{aligned}$$

We additionally derive, under the statistical significance framework [7] of normalizing the bin distributions (into the standard normal distribution), the associated peak finding probability which can be directly computed as an integral, given the desired statistical significance threshold  $S_{\text{th}}$ . This is given by

$$\text{Prob}[\text{peak} > S_{\text{th}}] = \int_0^\infty dz \int_{-\infty}^\infty dx N f(x) F(x)^{N-1} \cdot \phi(x - z - S_{\text{th}}).$$

### Appendix E: Peak tracking derivation

We perform active frequency compensation by estimating the clock frequency offset from the set of timestamps. The timing difference  $\tau_i := t_i - t'_i$ , between a timestamp pair  $\{t_i, t'_i\}$  from both parties, is continuously served by searching for photon pair detection events within a prescribed coincidence tracking window.

The frequency offset  $\Delta u$  is given by the ratio between measured elapsed time  $\Delta t_i := t_i - t_{i-1}$  with respect to some reference elapsed time, in this case the elapsed time measured by the other peer  $\Delta t'_i$ ,

$$\frac{\Delta t_i}{\Delta t'_i} = 1 + \Delta u$$

Rewriting in terms of the measured successive timing difference, we can estimate the frequency offset by measuring the rate of change in the timing difference, i.e.,

$$\Delta u_i = \frac{\tau_i - \tau_{i-1}}{\Delta t'_i}$$

Active frequency compensation is therefore achieved by performing a timing correction for each timestamp  $t_i$  using the estimated frequency offset,

$$t_i \rightarrow \Delta t_i(1 + \Delta u) + t_{i-1} = t_i + \Delta t_i \Delta u \quad (\text{E1})$$

where the overall frequency offset  $\Delta u$  accumulated is also given by  $\Delta u = \prod_0^i (1 + \Delta u_i) - 1$ .

The peak tracking is intrinsically noisy due to the signal being dominated by accidental coincidences. We use an exponential moving average filter given by,

$$\tau'_i = \alpha \tau_i + (1 - \alpha) \tau_{i-1}, \quad \tau'_0 = \tau_0,$$

which behaves like a low pass filter to smooth the signal.

Given a unit step impulse, the time it takes to reach  $1 - 1/e$  of the signal is associated with a time constant  $\beta$ ,

$$\alpha = 1 - \exp\left(-\frac{\overline{\Delta t}}{\beta}\right) \approx \frac{\overline{\Delta t}}{\beta}, \quad \beta \gg \overline{\Delta t}$$

after Taylor expansion of the exponential, where  $\overline{\Delta t}$  is the average separation time between consecutive timestamp events. The singles rate  $s$  in our experiment is 200 kcounts/s, so  $\overline{\Delta t} = 1/s = 5 \mu\text{s}$ . We set the time constant  $\beta = 50 \text{ ms}$ .

### Appendix F: Longer time scale measurement

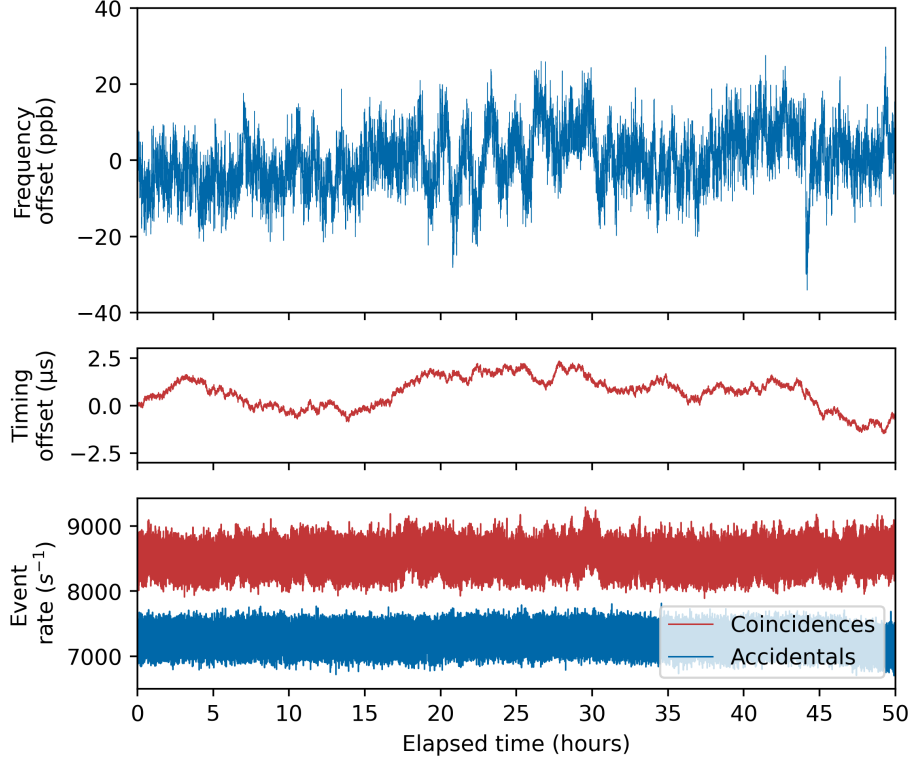


FIG. 7. Tracking measurement over 50 hours, similar to that of Fig. 1.

Fig. 7 shows a longer experimental measurement spanning over 50 hours, with a similar drift behaviour observed in the reconstructed frequency offset. The frequency offsets were not measured concurrently, so no tracking accuracy estimation is available for this dataset.

Measurement was performed using a different pair of Si APDs instead (Micro Photon Devices SPAD), which had an order of magnitude lower system efficiency and resulted in a lower  $-90$  dB attenuation channel possible at the same detection rate parameters. The measurement was also performed using suboptimal parameters to the peak tracking software `costream` that dropped off too many coincidences. The timestamps were onboard quartz clocks, so due to the heating of the clocks, the frequency offset between the two generally remained the same, and showed nearly an order of magnitude better frequency stability.

The measured frequency difference is low on the order of 10 ppb due to the fact that the two timestamps shared the same air conditioned environment, but this technique will still work for frequency drifts larger than 100 ppb.