Asymmetric delay attack on an entanglement-based bidirectional clock synchronization protocol – Supplementary Material

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In this supplementary material, we show that when circulators rotate the polarization state of one of the photons in an entangled pair by 180°, the geometric phase imposed on the rotated photon does not produce a measurable change in polarization entanglement.

We first introduce the formalism to deal with the fact that points on the Poincaré sphere carry no phase information; the beginning and end points of a cyclic evolution correspond on the same point on the sphere.

To reflect this property, we define a "basis vector field" $|\tilde{\psi}(t)\rangle$, such that

$$|\tilde{\psi}(t)\rangle = e^{-if(t)}|\psi(t)\rangle \quad \text{and} \quad |\tilde{\psi}(\tau)\rangle = |\tilde{\psi}(0)\rangle,$$

where $f(t)$ is the phase of $|\psi(t)\rangle$ expressed in terms of its basis state $|\tilde{\psi}(t)\rangle$ on the Poincaré sphere[30].

The change in f comprises of two terms

$$\Delta f = \beta + \gamma, \tag{S1}$$

where the geometric phase

$$\beta = \int_0^\tau \langle\tilde{\psi}(t)|i\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\psi}(t)\rangle \tag{S2}$$

is due to the evolution of the basis state along a curved geometry, and the dynamic phase

$$\gamma = -\int_0^\tau \langle\psi(t)|i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\,\mathrm{d}t \tag{S3}$$

is due to the photon's dynamics through the rotation medium[15].

A. Geometric Phase

Berry showed that the geometric phase is proportional only to the solid angle Ω subtended by the cyclic trajectory on the Poincaré sphere[16],

$$\beta = -\frac{1}{2}\Omega. \tag{S4}$$

Thus, a qubit in the initial state

$$|\psi(t=0)\rangle = e^{-i\phi}\cos(\theta/2)|R\rangle + \sin(\theta/2)|L\rangle \tag{S5}$$

that underwent a 180° rotation in the plane of polarization ($\phi \to \phi + 2\pi$) will accumulate a geometric phase $\beta = -\pi(1-\cos\theta)$.

B. Dynamic Phase

To evaluate the dynamic phase γ accumulated by the photon at end of a Faraday Rotator of length d, we parameterize its expression in Eq. S3 in terms of the penetration depth z

$$\gamma = -\int_0^d \langle\psi(z)|i\frac{\mathrm{d}}{\mathrm{d}z}|\psi(z)\rangle dz \quad = -\int_0^d \langle\psi(z)|\hat{N}|\psi(z)\rangle dz, \tag{S6}$$

where

$$\hat{N} = k\begin{pmatrix} n_R & 0 \\ 0 & n_L \end{pmatrix} \quad \text{and} \quad |\psi(z)\rangle = \begin{pmatrix} e^{ikn_R z}e^{-i\phi}\cos(\theta/2) \\ e^{ikn_L z}sin(\theta/2) \end{pmatrix} \tag{S7}$$

are expressed in the $\{|R\rangle, |L\rangle\}$ basis, and $k = \frac{2\pi}{\lambda}$ is the wave number of the photon mode in free space.

The Faraday Rotator is a birefringent medium whose refractive indices $n_{R,L}$ depend on the magnitude of an applied magnetic field B in the direction of light propagation,

$$n_{R,L} = n_0\left(1 \pm \frac{VB}{kn_0}\right), \tag{S8}$$

where V is the Verdet constant and n_0 is the index of refraction in the absence of a magnetic field.

Substituting S7 into S6, we obtain

$$\gamma = kn_0 d + VBd\cos\theta, \tag{S9}$$

where the product VBd can be shown[31] to be the anticlockwise rotation angle for a linearly polarized input.

Consider an initial input state $|\psi(\phi=0,\theta=0)\rangle = |H\rangle$. For the evolution cycle ($\phi = 0 \to 2\pi$) considered earlier, $|H\rangle \to |-45\rangle \to |V\rangle \to |+45\rangle \to |H\rangle$ corresponds to a *clockwise* 180$^{\text{o}}$ in the plane-of-polarization. Thus, the the rotation must be realized by a medium whose product $VBd = -\pi$. Consequently, the dynamic phase $\gamma = kn_0d - \pi\cos\theta$ for the state considered in Eq. S5.

C. Overall Phase & the Circulator Attack

We have already shown that an initial state

$$|\psi\rangle = \cos(\theta/2)|R\rangle + \sin(\theta/2)|L\rangle, \tag{S10}$$

will accumulate a geometric phase $\beta = -\pi(1-\cos\theta)$ and a dynamic phase $\gamma = kn_0d - \pi\cos\theta$, resulting in an overall phase $\phi = kn_0d - \pi$. Repeating this procedure for the orthogonal state

$$|\psi_\perp\rangle = -\sin(\theta/2)|R\rangle + \cos(\theta/2)|L\rangle, \tag{S11}$$

we obtain a geometric phase of $\beta' = -\beta = +\pi(1-\cos\theta)$ and a dynamic phase $\gamma' = kn_0d + \pi\cos\theta$, resulting in an overall phase $\phi' = kn_0d + \pi = \phi + 2\pi$.

Let the entangled pair initially be in the Bell state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$. With the first qubit Alice's photon and the second one Bob's photon. We can rewrite the Bell state in the basis defined by Equations S10 and S11,

$$\begin{aligned}|\Psi^-\rangle &= \frac{1}{\sqrt{2}}\Big(|HV\rangle - |VH\rangle\Big)\\ &= \frac{i}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B - |\psi\rangle_A|\psi_\perp\rangle_B\Big).\end{aligned} \tag{S12}$$

The state of the Bell pair after Bob's photon goes through Eve's circulator based attack, $\hat{U}_{Attack}$, is given by

$$\begin{aligned}|\Psi^-\rangle \to \hat{U}_{Attack}&\frac{i}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B - |\psi\rangle_A|\psi_\perp\rangle_B\Big)\\ &= \frac{i}{\sqrt{2}}\Big(e^{i\phi}|\psi_\perp\rangle_A|\psi\rangle_B - e^{i\phi'}|\psi\rangle_A|\psi_\perp\rangle_B\Big) \qquad \text{(S13)}\\ &= \frac{ie^{i\phi}}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B - e^{i2\pi}|\psi\rangle_A|\psi_\perp\rangle_B\Big)\\ &= e^{i\phi}|\Psi^-\rangle = -e^{ikn_0d}|\Psi^-\rangle\\ &\equiv -|\Psi^-\rangle. \qquad \text{(S14)}\end{aligned}$$

We can see from this expression, that the initial Bell state remains unchanged from the introduction of the circulators, and is equivalent to the result obtained by direct calculation in Eq. 5 in the main text.

Recent work wrongly assumed that the contribution from the dynamic phase was "zero, or is known and compensated for" and predicted instead that the circulators imparted a non-local geometric phase to produce a dramatic change[15]

$$\begin{aligned}|\Psi^-\rangle \to \hat{U}_{Attack}&\frac{i}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B - |\psi\rangle_A|\psi_\perp\rangle_B\Big)\\ &= \frac{i}{\sqrt{2}}\Big(e^{i\beta}|\psi_\perp\rangle_A|\psi\rangle_B - e^{-i\beta}|\psi\rangle_A|\psi_\perp\rangle_B\Big).\end{aligned} \tag{S15}$$

However, Eq. S9 shows that the dynamic phase is likewise non-local (due to its dependence on θ) and combines with the geometric phase to produce no measurable net change in the state.

We note that when geometric phases were observed in other entangled systems, an interferometric arrangement was necessary to eliminate the influence of this "dynamic" phase[17–20]. Whether or not a similar technique can be used to secure the present synchronization protocol remains an open question.