Asymmetric delay attack on an entanglement-based bidirectional clock synchronization protocol – Supplementary Material

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In this supplementary material, we show that when circulators rotate the polarization state of one of the photons in an entangled pair by 180°, the geometric phase imposed on the rotated photon does not produce a measurable change in polarization entanglement.

We first introduce the formalism to deal with the fact that points on the Poincaré sphere carry no phase information; the beginning and end points of a cyclic evolution correspond on the same point on the sphere.

To reflect this property, we define a "basis vector field" $|\tilde{\psi}(t)\rangle$, such that

$$|\tilde{\psi}(t)\rangle = e^{-if(t)}|\psi(t)\rangle \quad \text{and} \quad |\tilde{\psi}(\tau)\rangle = |\tilde{\psi}(0)\rangle,$$

where $f(t)$ is the phase of $|\psi(t)\rangle$ expressed in terms of its basis state $|\tilde{\psi}(t)\rangle$ on the Poincaré sphere[31].

The change in f comprises of two terms

$$\Delta f = \beta + \gamma, \tag{S1}$$

where the geometric phase

$$\beta = \int_0^\tau \langle\tilde{\psi}(t)|i\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\psi}(t)\rangle \tag{S2}$$

is due to the evolution of the basis state along a curved geometry, and the dynamic phase

$$\gamma = -\int_0^\tau \langle\psi(t)|i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle\,\mathrm{d}t \tag{S3}$$

is due to the photon's dynamics through the rotation medium[15].

A. Geometric Phase

Berry showed that the geometric phase is proportional only to the solid angle Ω subtended by the cyclic trajectory on the Poincaré sphere[16],

$$\beta = -\frac{1}{2}\Omega. \tag{S4}$$

Thus, a qubit in the initial state

$$|\psi(t=0)\rangle = e^{-i\phi}\cos(\theta/2)|R\rangle + \sin(\theta/2)|L\rangle \tag{S5}$$

that underwent a 180° rotation in the plane of polarization ($\phi \to \phi + 2\pi$) will accumulate a geometric phase $\beta = -\pi(1-\cos\theta)$.

B. Dynamic Phase

To evaluate the dynamic phase γ accumulated by the photon at end of a Faraday Rotator of length d, we parameterize its expression in Eq. S3 in terms of the penetration depth z

$$\gamma = -\int_0^d \langle\psi(z)|i\frac{\mathrm{d}}{\mathrm{d}z}|\psi(z)\rangle dz \quad = -\int_0^d \langle\psi(z)|\hat{N}|\psi(z)\rangle dz, \tag{S6}$$

where

$$\hat{N} = k\begin{pmatrix} n_R & 0 \\ 0 & n_L \end{pmatrix} \quad \text{and} \quad |\psi(z)\rangle = \begin{pmatrix} e^{ikn_R z}e^{-i\phi}\cos(\theta/2) \\ e^{ikn_L z}sin(\theta/2) \end{pmatrix} \tag{S7}$$

are expressed in the $\{|R\rangle, |L\rangle\}$ basis, and $k = \frac{2\pi}{\lambda}$ is the wave number of the photon mode in free space.

The Faraday Rotator is a birefringent medium whose refractive indices $n_{R,L}$ depend on the magnitude of an applied magnetic field B in the direction of light propagation,

$$n_{R,L} = n_0\left(1 \pm \frac{VB}{kn_0}\right), \tag{S8}$$

where V is the Verdet constant and n_0 is the index of refraction in the absence of a magnetic field.

Substituting S7 into S6, we obtain

$$\gamma = kn_0 d + VBd\cos\theta, \tag{S9}$$

where the product VBd can be shown[32] to be the anticlockwise rotation angle for a linearly polarized input.

Consider an initial input state $|\psi(\phi=0,\theta=0)\rangle=|H\rangle$. For the evolution cycle ($\phi=0\to 2\pi$) considered earlier, $|H\rangle\to|{-45}\rangle\to|V\rangle\to|{+45}\rangle\to|H\rangle$ corresponds to a *clockwise* 180^{o} in the plane-of-polarization. Thus, the the rotation must be realized by a medium whose product $VBd=-\pi$. Consequently, the dynamic phase $\gamma=kn_0d-\pi\cos\theta$ for the state considered in Eq. S5.

C. Overall Phase & the Circulator Attack

We have already shown that an initial state

$$|\psi\rangle=\cos(\theta/2)|R\rangle+\sin(\theta/2)|L\rangle, \tag{S10}$$

will accumulate a geometric phase $\beta=-\pi(1-\cos\theta)$ and a dynamic phase $\gamma=kn_0d-\pi\cos\theta$, resulting in an overall phase $\phi=kn_0d-\pi$. Repeating this procedure for the orthogonal state

$$|\psi_\perp\rangle=-\sin(\theta/2)|R\rangle+\cos(\theta/2)|L\rangle, \tag{S11}$$

we obtain a geometric phase of $\beta'=-\beta=+\pi(1-\cos\theta)$ and a dynamic phase $\gamma'=kn_0d+\pi\cos\theta$, resulting in an overall phase $\phi'=kn_0d+\pi=\phi+2\pi$.

Let the entangled pair initially be in the Bell state $|\Psi^-\rangle=\frac{1}{\sqrt{2}}(|HV\rangle-|VH\rangle)$. With the first qubit Alice's photon and the second one Bob's photon. We can rewrite the Bell state in the basis defined by Equations S10 and S11,

$$\begin{aligned}|\Psi^-\rangle&=\frac{1}{\sqrt{2}}\Big(|HV\rangle-|VH\rangle\Big)\\&=\frac{i}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B-|\psi\rangle_A|\psi_\perp\rangle_B\Big).\end{aligned} \tag{S12}$$

The state of the Bell pair after Bob's photon goes through Eve's circulator based attack, $\hat{U}_{Attack}$, is given by

$$\begin{aligned}|\Psi^-\rangle&\to\hat{U}_{Attack}\frac{i}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B-|\psi\rangle_A|\psi_\perp\rangle_B\Big)\\&=\frac{i}{\sqrt{2}}\Big(e^{i\phi}|\psi_\perp\rangle_A|\psi\rangle_B-e^{i\phi'}|\psi\rangle_A|\psi_\perp\rangle_B\Big) \qquad \text{(S13)}\\&=\frac{ie^{i\phi}}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B-e^{i2\pi}|\psi\rangle_A|\psi_\perp\rangle_B\Big)\\&=e^{i\phi}|\Psi^-\rangle=-e^{ikn_0d}|\Psi^-\rangle\\&\equiv-|\Psi^-\rangle. \qquad \text{(S14)}\end{aligned}$$

We can see from this expression, that the initial Bell state remains unchanged from the introduction of the circulators, and is equivalent to the result obtained by direct calculation in Eq. 5 in the main text.

Recent work assumed that the contribution from the dynamic phase was "zero, or is known and compensated for" and predicted instead that the circulators imparted a non-local geometric phase to produce a dramatic change[15]

$$\begin{aligned}|\Psi^-\rangle&\to\hat{U}_{Attack}\frac{i}{\sqrt{2}}\Big(|\psi_\perp\rangle_A|\psi\rangle_B-|\psi\rangle_A|\psi_\perp\rangle_B\Big)\\&=\frac{i}{\sqrt{2}}\Big(e^{i\beta}|\psi_\perp\rangle_A|\psi\rangle_B-e^{-i\beta}|\psi\rangle_A|\psi_\perp\rangle_B\Big).\end{aligned} \tag{S15}$$

However, we note that the dynamic phase (Eq. S9) is likewise non-local (due to its dependence on θ) and combines with the geometric phase to produce no measurable net change in the state.