Coupling Light to Higher Order Transverse Modes of a Near-Concentric Optical Cavity: Supplemental Document

In this supplemental document, we provide a more thorough calculation of the atom-cavity coupling constant g_{ac} for radial transverse modes (LG modes with $l = 0$). We show that, at a particular cavity length L or critical distance d, all the radial transverse modes have the same mode volume V_m and hence the same coupling constant g_{ac}. This would allow an atom to have similar interaction strengths with all of the radial transverse modes of the cavity.

1. INTERACTION HAMILTONIAN OF AN ATOM-CAVITY SYSTEM

An electric field operator with a complex mode amplitude profile $U_j(\mathbf{r})$ for configuration parameter j can be expressed as [1]

$$\hat{\mathbf{E}}_j(\mathbf{r}) = i\mathbf{e}_\lambda N_j \left[U_j(\mathbf{r})\hat{a}_j - U_j^*(\mathbf{r})\hat{a}_j^\dagger \right], \tag{S1}$$

where $\mathbf{e}_\lambda$ is the electric field polarization unit vector, N_j is a normalization coefficient, and $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the destruction and creation mode operators. The free field Hamiltonian of the electromagnetic mode inside a cavity with volume V has a similar structure with a harmonic oscillator,

$$\hat{H}_0 = \frac{\epsilon_0}{2}\int dV \left[\hat{\mathbf{E}}_j(\mathbf{r})^2 + c^2\hat{\mathbf{B}}_j(\mathbf{r})^2 \right] = \hbar\omega_j \left(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2} \right), \tag{S2}$$

with $\hat{\mathbf{B}}_j(\mathbf{r})$ as the corresponding magnetic field operator. By imposing the normalization condition

$$\int dV |U_j(\mathbf{r})|^2 = 1\,, \tag{S3}$$

we obtain the normalization coefficient $N_j = (\hbar\omega_j/2\epsilon_0)^{1/2}$.

The interaction Hamitonian between an electromagnetic field and an atomic dipole, in the electric-dipole approximation, is given by

$$\hat{H}_I(\mathbf{r}) = -\hat{\mathbf{E}}_j(\mathbf{r}) \cdot \hat{\mathbf{d}}\,, \tag{S4}$$

where $\hat{\mathbf{d}} = d_a \left(\hat{\pi}^\dagger + \hat{\pi} \right) \mathbf{e}_d$ is the atomic electric-dipole operator with d_a as the dipole moment, $\hat{\pi}^\dagger = |e\rangle\langle g|$ and $\hat{\pi} = |g\rangle\langle e|$ as the transition operators between ground and excited atomic states, and $\mathbf{e}_d$ as the dipole unit vector. Assuming that the atomic dipole and electric field polarization vectors point to the same direction, $\mathbf{e}_\lambda = \mathbf{e}_d$, we obtain

$$\hat{H}_I(\mathbf{r}) = -id_a\sqrt{\frac{\hbar\omega_j}{2\epsilon_0}} \left[U_j(\mathbf{r})\hat{a}_j - U_j^*(\mathbf{r})\hat{a}_j^\dagger \right] \left[\hat{\pi}^\dagger + \hat{\pi} \right], \tag{S5}$$

with an explicit dependence of the atom position $\mathbf{r}$ on the mode amplitude $U_j(\mathbf{r})$.

2. ATOM-CAVITY COUPLING CONSTANT FOR RADIAL TRANSVERSE MODES

The complex amplitude of radial transverse modes can be obtained by setting $l = 0$ to the LG mode amplitude (Eq. 1 of the main text),

$$U_{p,0}(\rho,\phi,z) = A_{p,0}\frac{w_0}{w(z)}\mathcal{L}_p^0\left(\frac{2\rho^2}{w^2(z)}\right)\exp\left(-\frac{\rho^2}{w^2(z)}\right)\exp\left(i\psi_{p,0}(\rho,\phi,z)\right). \tag{S6}$$

The radial transverse modes of a near-concentric cavity have spatial dependencies on both the axial and radial direction [2]. Here, we assume that the atom is located at the center of the cavity

mode ($\mathbf{r} = \mathbf{0}$), such that $\rho = 0$ and $z = 0$. The amplitude of the radial transverse mode at this center point is

$$U_{p,0}(\mathbf{r} = \mathbf{0}) = A_{p,0}\,, \tag{S7}$$

where we have evaluated $w(0) = w_0$, $\mathcal{L}_p^0(0) = 1$, and $\exp\left(i\psi_{p,0}(0,\phi,0)\right) = 1$.

The prefactor $A_{p,0}$ can be determined from the normalization condition (Eq. S3), and has a physical significance as follows. If we assume a theoretical cavity with a uniform distribution of mode amplitude $U_{p,0} = A_{p,0}$, the normalization condition yields

$$A_{p,0} = \frac{1}{\sqrt{V_{m,p}}}\,, \tag{S8}$$

where $V_{m,p}$ is the mode volume of such theoretical cavity.

Using Eq. S5, we evaluate the interaction Hamiltonian of an atom placed at the centre of a radial transverse cavity mode,

$$\hat{H}_I(\mathbf{r} = \mathbf{0}) = -i\hbar g_{ac}\left[\hat{a}_j - \hat{a}_j^\dagger\right]\left[\hat{\pi}^\dagger + \hat{\pi}\right]\,, \tag{S9}$$

with the atom-cavity coupling constant

$$g_{ac} = \sqrt{\frac{\omega_j d_a^2}{2\hbar\epsilon_0 V_{m,p}}}\,. \tag{S10}$$

It is interesting to note that the coupling constant $g_{ac} \propto 1/\sqrt{V_{m,p}}$ only depends on the radial mode number p through the mode volume $V_{m,p}$. This is not necessarily true for other classes of transverse modes. For example, non-radial transverse modes ($l \neq 0$) has zero mode amplitude at the centre, i.e. $U_{p,l\neq 0}(\mathbf{r} = \mathbf{0}) = 0$, which complicates the interpretation of such mode volumes.

3. MODE VOLUME CALCULATION OF RADIAL TRANSVERSE MODES

In this part, we show that the mode volume of radial transverse modes $V_{m,p}$ only depends on the beam waist w_0 and cavity length L, and does not depend on the radial mode number p. Using the normalization condition (Eq. S3),

$$\int dV |U_{p,0}(\rho,\phi,z)|^2 = \int dz \int \rho d\rho \int d\phi\, \frac{1}{V_{m,p}}\left[\frac{w_0}{w(z)}\mathcal{L}_p^0\left(\frac{2\rho^2}{w^2(z)}\right)\right]^2 \exp\left(-\frac{2\rho^2}{w^2(z)}\right) \tag{S11}$$

$$1 = \frac{1}{V_{m,p}}\frac{\pi w_0^2 L}{2}\int_0^\infty du\, e^{-u}\mathcal{L}_p^0(u)^2\,, \tag{S12}$$

where we have used the substitution $u = 2\rho^2/w^2(z)$. The integration can be calculated using the orthonormal property of Laguerre polynomials [3],

$$\int_0^\infty dx\, e^{-x}\mathcal{L}_m^0(x)\mathcal{L}_n^0(x) = \delta_{m,n}\,. \tag{S13}$$

The mode volume of radial transverse modes is thus calculated to be

$$V_{m,p} = \frac{1}{2}\pi w_0^2 L\,, \tag{S14}$$

independent of the radial mode number p. At a particular cavity length L, the beam waist parameter w_0 is evaluated from the cavity boundary conditions, and is also independent on the radial mode number p. Hence, the atom-cavity coupling constant g_{ac} is maintained over all the radial transverse modes,

$$g_{ac} = \sqrt{\frac{\omega_j d_a^2}{\pi\hbar\epsilon_0 w_0^2 L}}\,. \tag{S15}$$

As a completing remark, we note that the mode of a physical cavity forms a standing wave on the longitudinal direction. Hence, the complex term $\exp(i\psi_{p,0})$ in Eq. S6 should be replaced with $\cos(\psi_{p,0} + \delta)$ with δ determined by the boundary condition and longitudinal mode number. Assuming the atom is located at the anti-node of such standing wave, the evaluation of the atom-cavity coupling is similar as above, but with the mode volume reduced to half,

$$V_{m,p}^{sw} = \frac{1}{4}\pi w_0^2 L\,. \tag{S16}$$

REFERENCES

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3. G. B. Arfken and H. J. Weber, "Mathematical methods for physicists," (1999).