Error analysis in the measurement of the CHSH inequality

COUNTING STATISTICS

The SPDC process produces photon pairs at random times. We can then assume that the poissonian statistics well represents the probability distribution of the count within a finite time interval, with a mean value equal to the number of detected coincidences N.

$$\Delta N_p = \sqrt{N} \tag{1}$$

TIME UNCERTAINTIES

Time uncertaineties affect the total acquisition time T. The uncertainty on the number of counts in a finite time interval N is proportional to the time uncertainty times the count rate R, defined as $R = N/T$.

$$\Delta N_T = R \cdot \Delta T = N \frac{\Delta T}{T}. \tag{2}$$

DETECTOR DEAD TIME

The effect of the dead time of the detector T_D is to reduce the effective measurement time T by a quantity $S_{1,2} \cdot T_D$, with $S_{A,B}$ representing the total number of single counts for the detectors A, B. The uncertainty in the total measurement time is then proportional to the uncertainty in the single counts: $\Delta T = T_D \cdot \Delta S_{A,B} = T_D \sqrt{S_{A,B}}$. Applying Eq. (2)), we obtain:

$$\Delta N_D = N \sqrt{(S_A + S_B)} \frac{T_D}{T}. \tag{3}$$

POLARIZERS

In order to assess how the uncertainty in the angular position of the polarizers propagates into an uncertainty in N, we assume that the relation between N, θ_A, and θ_B is well described by standard quantum mechanics. If this is the case, for the singlet state we expect N to be a function of the difference $\theta = \theta_A - \theta_B$: $N = \frac{1}{2}(sin\theta)^2$. We can easily then propagate the uncertainty $\Delta\theta_A = \Delta\theta_B = \Delta\theta$ to obtain:

$$\Delta N_R = \sqrt{2} N \cot\theta \, \Delta\theta. \tag{4}$$

TOTAL ERROR ON N

We can combine the errors expressed by Eqs. (1,2,3,4)) to calculate the uncertainty associated with each measurement N:

$$\Delta N = \sqrt{(\Delta N_P)^2 + (\Delta N_T)^2 + (\Delta N_D)^2 + (\Delta N_R)^2}, \tag{5}$$

where each term is calculated for the corresponding angular setting θ and also include the single counts S_A and S_B.

PROPAGATION OF THE ERROR TO E

In order to propagate the error to E, we use the standard propagation for the statistical error:

$$\Delta E^2 = \sum_{i=1}^{4} \left(\frac{\partial E}{\partial N_i} \right)^2 \Delta N_i^2 \tag{6}$$

from the definition of E, we derive:

$$\frac{\partial E}{\partial N_i} = \begin{cases} 2 \frac{N_3 + N_4}{\left(\sum_{i=1}^{4} N_i \right)^2} & \text{for } i = 1, 2 \\ 2 \frac{N_1 + N_2}{\left(\sum_{i=1}^{4} N_i \right)^2} & \text{for } i = 3, 4 \, , \end{cases} \tag{7}$$

from which it follows:

$$\Delta E = \frac{2 \left[(N_3 + N_4)^2 (\Delta N_1^2 + \Delta N_2^2) + (N_1 + N_2)^2 (\Delta N_3^2 + \Delta N_4^2) \right]}{\left(\sum_{i=1}^{4} N_i \right)^2} \tag{8}$$

We use the Matlab script `err_estimation.m` to numerically propagate the error from the the 16 Ns to the final S. It is possible to derive a single analytical expression but it would contain $16 * 3$ variables: the 16 coincidences and the 16 singles counts from A and B

On the other hand, with some basic assumption it is possible to figure out the dependencies.

Assuming that $N_i = N \cdot c_i$, with $N = \sum_i N_i$, $E = \frac{1}{\sqrt{2}}$ and that N and ΔN are the same for all measurements, we obtain:

$$\Delta E = \frac{\Delta N}{2N}. \tag{9}$$

Under the same assumptions, we obtain the uncertainty for S:

$$\Delta S = \frac{\Delta N}{N}. \tag{10}$$

Replacing ΔN with the uncertainty associated only with ΔT, we obtain the scaling:

$$\Delta S_T = \frac{\Delta T}{T}. \tag{11}$$